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WITH SECTIONS ON BOOK-KEEPING, DECIMAL COINAGE,
AND THE METRICAL SYSTEM.

BY THE

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THE FOLLOWING SIGNS MUST BE VERY WELL KNOWN BEFORE THIS BOOK IS READ.

+ *plus*, placed between two numbers, shows that they are to be added together.

— *minus*, between two numbers, shows that the latter number is to be subtracted from the former.

× *into*, or *multiplied by*, between two numbers, shows that the two are to be multiplied together.

÷ *by*, or *divided by*, between two numbers, shows that the former is to be divided by the latter.

= *equal to*, between two quantities, shows that the quantities on each side of it are equal.

∴ stands for *therefore*.

∵ *because*, or *since*.

> *greater than*, placed between two quantities, shows that the former is greater than the latter.

< *less than*, placed between two quantities, shows that the former is less than the latter.

∼ placed between two quantities, shows that the smaller of the two is to be subtracted from the larger.

Note.—The figures enclosed in parentheses, as (8) in p. 4, line 22, refer to previous articles in the work.

ARITHMETIC.

Before commencing the consideration of Fractions, it will be necessary to explain certain properties of numbers, which enter very much into the treatment of fractions.

Errata.

Page 226, line 19, the words *Cr.* and *Dr.* should be reversed.
„ 239, in the middle of the page, „ „
„ 240, lines 1 and 3, „ „
„ 235, erase the note.

of 6, 4, and 2, because it can be divided exactly by those numbers; and 12 is the *least* common multiple of 6, 4, and 2, because it is the *least* number that can be divided exactly by those three divisors.

3. A number is said to be a common measure of two or more numbers when it will exactly divide those numbers; and the greatest number which will so divide them is called

their greatest common measure. Thus, 2, 4, 6, 12, are common measures of 24 and 36; and 12 is their greatest common measure.

We shall for the future write G. C. M. for greatest common measure, and L. C. M. for least common multiple.

4. A number is called a *prime* number when it cannot be divided exactly by any number greater than 1. Thus, 3, 5, 7, are prime numbers, because they have no divisor greater than 1; we also call such numbers *primes.*

5. Two or more numbers are said to be prime to one another, when they cannot *all* be divided exactly by any number greater than 1. Thus, 7, 8, 9, are prime to each other: but such numbers are not necessarily themselves prime numbers; for 8 can be divided by 2 and 4; and 9 can be divided by 3; but 8 and 9 have no divisor common to both.

6. All numbers not prime are called *composite;* because they are composed of two or more prime numbers multiplied together. Thus, 6 is a composite number, being formed by the multiplication of the prime numbers 2 and 3.

7. The prime numbers which, when multiplied together, form a composite number, are called *factors* of that number. Thus, 2, 2, 2, 3, are factors of 24, because $2 \times 2 \times 2 \times 3 = 24$. So the factors of 36 are 2, 2, 3, 3, for $2 \times 2 \times 3 \times 3 = 36$.

8. To break up a composite number into its prime factors, divide it by the smallest prime which will divide it without remainder; and continue dividing by prime divisors until the last quotient is 1. The number will be found equal to the product of all the divisors—that is, all the divisors are its factors.

Ex. To break 120 into prime factors:

$$\begin{array}{r|r} 2 & 120 \\ \hline 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

∴ 2, 2, 2, 3, 5, are its prime factors,
or, $120 = 2 \times 2 \times 2 \times 3 \times 5$.

When, therefore, I say—break any number, as 120, into its factors, I mean, exhibit it in the above form. This form is called an equation: thus, $7 \times 5 = 35$ is an equation; and 7×5 is called the left-hand *side*, and 35 the right-hand *side*, of the equation.

Exs. 1. Resolve into prime factors

1. 1050.
2. 2625.
3. 1485.
4. 1155.
5. 1820.
6. 4802.
7. 64.
8. 4389.
9. 2310.
10. 6342.
11. 5724.
12. 1168.

Obs. If any one number be divisible by a second, we must have all the factors of the second number among the factors of the first: thus, if 36 be divisible by 12, all the factors of 12 will be among the factors of 36, or 36 must contain all the factors of 12.

9. Also, if I wish to divide 35 by 5, (since 35 is the same as 7×5), I may divide 7×5 by 5; and since I know that the quotient is 7, I can now observe that that quotient is found by taking away the divisor 5 out of the product 7×5: so also in the equation $7 \times 3 \times 5 = 105$, to divide the 105 by 5, take away the 5 out of the factors on the left-hand side, and there remains 7×3, or 21, as quotient; to divide by 7, take away 7, and 3×5, or 15, is quotient; or, we can divide by the product of two factors at once: thus, to divide by 15, take away 3×5, and the quotient is 7. This is a very quick method of dividing a composite number by any one or more of its factors, especially if the number be large.

Thus, since $2520 = 2 \times 2 \times 2 \times 7 \times 3 \times 3 \times 5$;

or $= 8 \times 7 \times 9 \times 5$;

therefore, if I wish to divide 2520 by any measure of it, as 24, I take away 24, or $2 \times 2 \times 2 \times 3$, from its factors, and the product of the remaining factors, $7 \times 3 \times 5 = 105$, gives the quotient. To divide 2520 by 63, I remove 7×9, and the quotient $= 8 \times 5 = 40$.

LEAST COMMON MULTIPLE.

10. It is required to find the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20.

Now, of the above numbers, we observe that 2, 3, 6, and 9 are divisors, or measures, of 18; every number, therefore, which is a multiple of 18, will also be a multiple of 2, 3, 6, 9. So likewise, every multiple of 20 will be a multiple of 5 and 10; therefore, every multiple of 18 and of 20 will be a multiple of 2, 3, 6, 9, 5 and 10. If, then, I find the L. C. M. of the three remaining numbers 12, 18, and 20, that multiple will be the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20. Now, $12 \times 18 \times 20$, or 4320 gives *one* common multiple of 12, 18, and 20, since this product is plainly divisible by 12, 18, and 20; but this is not the *least* C. M.

11. To find the L. C. M., break the numbers 12, 18, 20, into their prime factors, as in (8), and place a comma between each set of factors, thus:

12,	18,	20,
$2 \times 2 \times 3$,	$2 \times 3 \times 3$,	$2 \times 2 \times 5$.

Now, by (8 OBS.), in order that the L. C. M. may contain 12, it must contain all the factors of 12; ∴ it must contain 2, 2, 3,

so, to contain 18, ... 2, 3, 3,

and to contain 20, ... 2, 2, 5;

that is, in the required L. C. M. I want two twos, two threes, and one 5; but in the product of $12 \times 18 \times 20$, or of $2 \times 2 \times 3 \times 2 \times 3 \times 3 \times 2 \times 2 \times 5$, I have five twos and three threes; therefore I have three twos and one 3 which I do not require; and, omitting these, the remaining factors, multiplied together, give $2 \times 2 \times 3 \times 3 \times 5 = 180$; where it may be seen upon trial that this row of factors contains all that is necessary for 12, 18, and 20, and no more; and therefore the product of these factors is the least number divisible by 12, 18, and 20, or is their L. C. M.

12. The whole work should be written out as follows, where the mark (–) is placed over every number which is afterwards to be omitted:—

$$\bar{2},\ \bar{3},\ \bar{5},\ \bar{6},\ \bar{9},\ \overline{10},\ 12,\ 18,\ 20.$$

$$2 \times 2 \times \bar{3}, \qquad \bar{2} \times 3 \times 3, \qquad \bar{2} \times \bar{2} \times 5,$$

$$\therefore \text{L. C. M.} = 4 \times 9 \times 5 = 20 \times 9 = 180 \ldots\ldots\ldots\ldots (A).$$

OBS. It is worth notice that, in multiplying the numbers which are to form the L. C. M., a little dexterity will generally enable the pupil to find the product mentally; for instance, the numbers very often contain one quantity ending in 5, and one even number; and these can be more readily multiplied than any others, as is seen in line (A).

Ex. II. To find the L. C. M. of 7, 16, 32, 21, 56, 42.

$$\bar{7},\ \overline{16},\ 32,\ \overline{21},\ 56,\ 42,$$

$$2 \times 2 \times 2 \times 2 \times 2, \qquad \bar{2} \times \bar{2} \times \bar{2} \times \bar{7}, \qquad \bar{2} \times 3 \times 7,$$

$$\therefore \text{L. C. M.} = 32 \times 21 = 672.$$

In this example, I first reject 7 and 21, because 42 contains them; next I reject 16, because 32 contains it; then, of the prime factors, I preserve five twos for the number 32, and reject the others: I preserve one 7, which is required for 56 and 42, and one 3 for 42. In preserving the five twos, it is better to keep them all together, rather than have some in one set of factors, and some in another.

Exs. 2. **Find the L. C. M. of**

1. 2, 3, 4, 8.
2. 3, 5, 9, 16, 20.
3. 3, 11, 5, 35, 21.
4. 7, 28, 35, 42, 63.
5. 4, 5, 8, 24, 40, 120.
6. 5, 7, 9, 12, 15.
7. 13, 14, 56, 63, 72.
8. 1, 2, 3, 4, 5, 6, 7, 8, 9.
9. 12, 33, 55, 27, 18.
10. 7, 11, 13, 17.
11. 8, 9, 10, 11, 12, 15.
12. 10, 14, 21, 28, 35.
13. 18, 20, 24, 36, 48.
14. 27, 36, 45, 42, 16.

GREATEST COMMON MEASURE.

13. We have seen in (3) that the G. C. M. of two or more numbers is the largest divisor which can exactly be contained in those numbers.

Where the numbers are not large, this G. C. M. may be found by breaking them into their prime factors; and if any factors are contained in all the numbers, the product of these will be the G. C. M.

Thus, to find the G. C. M. of 36, 27, 144; breaking up into factors, we have

$$\begin{array}{ccc} 36, & 27, & 144, \\ 2 \times 2 \times 3 \times 3, & 3 \times 3 \times 3, & 2 \times 2 \times 2 \times 2 \times 3 \times 3, \end{array}$$

Here it is plain that in these sets of factors two threes and no other number are common to all; that is, 3 × 3, or 9, is common to all the numbers 36, 27, 144,—and is therefore their G. C. M.

14. But where the numbers are large, and they cannot be easily broken into factors, the following rule is to be used:—

Take two of the proposed numbers, and divide greater by the less: if there be a remainder, make that remainder a new divisor, and take the former divisor as a new dividend, and continue this process until there be no remainder: *the last divisor will be the* G. C. M.

If there be a third number, go through the same work with this third number, and the G. C. M. of the other two; then, as before, the last divisor will be the G. C. M. And the same process must be continued if there be 4, 5, &c., or any amount of numbers, of which we have to find the G. C. M.

We will try this rule upon the three numbers taken above, viz. 36, 27, and 144.

```
27) 36 (1              9) 144 (16
    27                    9
    --                    --
     9) 27 (3             54
        27                54
        --                --
```

∴ 9 is the G. C. M. of 27 and 36: and 9 is the G. C. M. of 27, 36, and 144, as was shewn above.

Ex. II. Find the G. C. M. of 324, 456, 728.

```
324) 456 (1                      12) 728 (60
     324                             720
     ---                             ---
     132) 324 (2                       8) 12 (1
          264                              8
          ---                              -
           60) 132 (2                      4) 8 (2
               120                            8
               ---                            -
                12) 60 (5
                    60
                    --
```

∴ 12 is the G. C. M. of 324, 456; and 4 is the G. C. M. of 324, 456, 728.

15. If the last divisor be 1, we then learn that the numbers have no common divisor greater than 1; *i. e.* they are prime to each other.

Obs. Even when the divisors are less than 12, it is better to divide by long, rather than short division, because the remainders are thereby placed in the most convenient situation for continuing the process. This rule for finding the G. C. M. cannot be proved true without the use of algebra; but the *method* of proof is shewn below[*].

16. When we have to find the L. C. M. or G. C. M. of any numbers which are not large, we may often, after a little experience, *see* the answer, without going through the work.

Thus, if it were required to find the L. C. M. of 3, 4, 6, and 8, a pupil would soon learn that 24 was the required number. The easiest method of performing this operation, without writing, is to multiply in one's head the largest of the numbers given, by 2, 3, 4, and so on, until a number be found which will contain all the other numbers. So, in 3, 4, 6, and 8, if I try $2 \times 8 = 16$, it will not hold the 6; but $3 \times 8 = 24$ will hold the 6 as well as the 3 and 4, and therefore is the required L. C. M.

17. Again, to find the G. C. M. of 4, 8, 12: it is easy to *see* that no number greater than 4 can go in 4, 8, and 12, or that 4 is their G. C. M.

When we can thus *see* the answer without working, we are said to find the L. C. M. or G. C. M. by *inspection*.

---

[*] When any number measures two others, it will be found to measure any number composed either of the *sum* of any multiples of those numbers, or their *difference*.

For example, since 6 is a common measure of 18 and 24, it will be a measure of $(3 \times 18) + (4 \times 24)$ which $= 150$; or of $(7 \times 18) - (3 \times 24)$ which $= 54$. This rule can be *proved* true by algebra, and may be seen to be correct in the particular Ex. just given.

Making use of this fact, and referring to the first portion of Ex. II., I observe that every C.M. of 324 and 456 must be a measure of $(456 - 324)$ or of 132; also that every C.M. of 132 and 324 must be a measure of $(132 + 324)$, or of 456; hence the common measures of 324 and 456 are precisely the same as those of 132 and 324; and therefore the *greatest* C.M. of 324 and 456 is the same as the *greatest* C.M. of 132 and 324. Again, I see that 60 and 132 are derived from 132 and 324, just as these latter numbers were derived from 324 and 456; hence it is plain that the G.C.M. of 132 and 324 is also the G.C.M. of 60 and 132, and still further, that it is the same as the G.C.M. of 12 and 60; hence the G.C.M. of the original numbers 324 and 456 is shewn to be the same as that of 12 and 60; but the G.C.M. of 12 and 60 is 12; hence 12 is the G.C.M. of 324 and 456.

Exs. 3. Find the G. C. M. of

1. 348 and 390.
2. 510 „ 595.
3. 413 „ 343.
4. 217 „ 643.
5. 1836 and 1845.
6. 4775 „ 10959.
7. 715, 781, and 1067.
8. 189, 216, „ 729.

Shew that 327 and 529, also that 189 and 727 are prime to each other.

OBS.—THE FOLLOWING ABBREVIATIONS WILL SOMETIMES BE USED.

Den^{r}	for	Denominator.	Rem^{r}	for	Remainder.
Num^{r}	„	Numerator.	Diff^{e}	„	Difference.
Fr^{n}	„	Fraction.	Comp^{d}	„	Compound.
Mult^{n}	„	Multiplication.	Red^{d}	„	Reduced.
Add^{n}	„	Addition.	Com.	„	Common.
Sub^{n}	„	Subtraction.	Quan^{y}	„	Quantity.
Div^{n}	„	Division.	Imp^{r}	„	Improper.
Quot^{t}	„	Quotient.	Frac^{l}	„	Fractional.
Ex.	„	Example.	Art.	„	Article.

FRACTIONS.

18. DEF. By the term *Unit*, we are to understand a single article of any kind, as 1 inch, 1 yard, 1 penny, 1 ounce, &c.

DEF. *Unity* is merely another name for the figure 1.

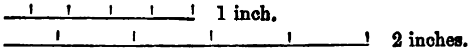
19. A fraction is a part or parts of a number, or quantity, supposed to be broken into any number of equal portions. If, then, the *unit* be divided into 4, 5, or 6 equal parts, one of these parts will be called one-fourth, one-fifth, or one-sixth, and is thus written—$\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$. So also, if any *two* of these parts be taken, the quantities thus taken will be called *two*-fourths, *two*-fifths, and *two*-sixths, *i. e.* of the unit or of *one*, and be written $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$.

20. The number below the line, which shews into how many parts the *unit* was broken, is called the *denominator*, because it expresses the denomination, or kind of parts, as *fourths*, *fifths*, *sixths*. The upper number, which *enumerates*, or counts, how many parts are taken, is called the *numerator*. Thus, in the fraction $\frac{2}{5}$, 5 is the denominator, 2 the numerator.

OBS. Such a fraction as I have been describing is called a *Vulgar Fraction*. We shall afterwards find that *Vulgar* Fractions can be expressed in another form, and are then called *Decimal* Fractions, or more commonly *Decimals*.

21. Again, any fraction, as $\frac{2}{5}$, may represent other quantities, besides being two-fifths of *one*, as we have just explained.

FIG. 1.



For if we divide the quantity 1, as 1 inch, into 5 equal parts (see Fig. 1), and also divide the quantity 2 into 5 equal parts, then one of the *latter* parts will be twice as great as one of the *former*; or *two* of the former and *one* of the latter have the same value. Now, the former or smaller is one-fifth of 1; and the latter or larger is one-fifth of 2; and, since two of the former = one of the latter, therefore two-fifths of 1 = one-fifth of 2, or $\frac{2}{5}$ of $1 = \frac{1}{5}$ of 2; and both these quantities are expressed by the fraction $\frac{2}{5}$. Similarly, $\frac{6}{7}$ means *six*-sevenths of *one*, or *one*-seventh of *six*; $\frac{3}{4}$*s.* means three-fourths of 1 shilling, or one-fourth of 3 shillings, each of which will, on trial, be found to be 9*d.*

22. From what has been said, it will be seen that the den^{r} of a fraction shews into how many parts the unit is divided; since, then, $\frac{6}{7}$ is proved to mean one-seventh of 6,

that is, it is equal to 6 divided by 7; we hence see that a number placed as a den^r, underneath any other number as a num^r, shews that this den^r is to be taken as a divisor of the num^r; thus, $\frac{15}{7}$ implies that 15 is to be divided by 7: but as long as I do not work out the div^n, that division is said only to be *expressed.* Thus again, in the fraction $\frac{23}{7}$, we understand that 23 *is to be divided* by 7.

We shall hereafter see more clearly than now, that if 23 is divided by 7, the fraction $\frac{23}{7}$ may be called the *quotient.*—(See Appendix, Art. *Fractional Quotient.*)

23. A fraction is called a *proper* fraction when the num^r is less than the den^r; thus, $\frac{3}{8}$, $\frac{4}{5}$, are called *proper* fractions, because they really represent a part or parts of a unit, and are less than the whole unit.

24. A fraction whose num^r is equal to, or is greater than the denom^r, is called an *improper* fraction: Thus, $\frac{5}{5}$, $\frac{12}{7}$, are called *improper* fractions, because they are not in reality *parts* of a unit broken up—*i. e.* are not less than the whole unit—but they are either one complete unit, or more than one.

25. Any whole number may be made to appear as an improper fraction with any required den^r, by multiplying and dividing the whole number by that denominator. Thus, $3=\frac{3\times 5}{5}=\frac{15}{5}$: if the required den^r be 7, $3=\frac{21}{7}$.

26. A mixed number is one formed of a whole number and a fraction, as $3\frac{2}{5}$, which is read three and two-fifths—that is, three units, and two-fifths of another unit; just as 3*s.* 7*d.* means 3 shillings and $\frac{7}{12}$ of another shilling, and might be written $3\frac{7}{12}$*s.*, where the unit is one shilling.

27. A fraction consisting of two or more fractions,

connected by the word *of* placed between them, is called a *compound* fraction; as $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{8}{9}$: and a *complex* fraction is one in which either the num^r or den^r, or both, are fractions; as

$$\frac{2\frac{3}{4}}{4},\qquad \frac{3}{7\frac{1}{2}},\qquad \frac{1\frac{1}{3}}{3\frac{3}{5}},\qquad \frac{\frac{8}{9}\text{ of }1\frac{2}{3}}{7\frac{1}{5}},\ \&c.$$

This last fraction is thus read: eight-ninths of one-and-two-thirds, divided by seven-and-one-fifth.

28. To multiply a fraction by any whole number, we multiply the num^r and retain the same den^r.

Thus, if it be required to multiply $\frac{5}{12}$ by 4,—just as 4 times 5 *pence* = 20 *pence*, so 4 times *5-twelfths* = 20-*twelfths*, or $\frac{20}{12}$;

$$\therefore\ \frac{5}{12}\times 4=\frac{20}{12},\ \text{which}=\frac{5\times 4}{12};\ \text{hence,}$$

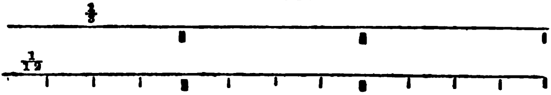
To multiply a fraction by a whole number, the num^r is multiplied by the number, and the den^r is not altered.

29. Again, to multiply a fraction by any whole number, *if possible*, divide the den^r by this multiplier, and leave the numerator unaltered. By this rule we should have

$$\frac{5}{12}\times 4=\frac{5}{12\div 4}=\frac{5}{3}.$$

To prove this, let any unit be divided—1st, into 3 equal parts, and 2ndly, into 12 equal parts (see Fig. 2); then

FIG. 2.



each of the larger parts is called $\frac{1}{3}$, and each of the smaller $\frac{1}{12}$. Now, one of the *thirds* is four times as large as one of the *twelfths*, therefore 5 of such thirds are 4 times as large as 5 of the twelfths;

$$\therefore \frac{5}{3} = 4 \text{ times } \frac{5}{12} = 4 \times \frac{5}{12};$$

or, in the former shape, $\frac{5}{12} \times 4 = \frac{5}{3}$, which $= \frac{5}{12 \div 4}$; hence,

In multiplying by a whole number, the den^r must be divided, and the num^r left unaltered.

I subjoin a specimen of the form in which these Exs. should be worked.

$$\frac{5}{18} \times 6 = \frac{5}{18 \div 6} = \frac{5}{3}.$$

$$\frac{5}{18} \times 8 = \frac{5 \times 8}{18} = \frac{40}{18}.$$

Exs. 4.

I. Multiply $\frac{5}{6}$ by 2, 3, 4, 5, 6, 7, 8, 9, successively.

II. Multiply $\frac{19}{24}$ by 3, 5, 6, 8, 9, 12, successively.

30. To divide a fraction by any whole number, divide the num^r, if possible, and keep the same den^r.

Thus, if it be required to divide $\frac{12}{13}$ by 4, just as 12 *pence* divided by 4 would give a quot^t 3 *pence*, so 12 *thirteenths* divided by 4 would give a quot^t 3 *thirteenths;*

$$\therefore \frac{12}{13} \div 4 = \frac{3}{13} \text{ which} = \frac{12 \div 4}{13}; \text{ hence,}$$

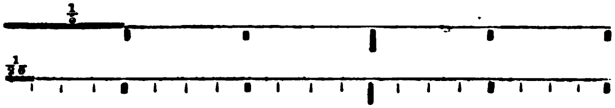
The num^r is divided by the given divisor, and the den^r unaltered.

But if we cannot divide the num^r of the fr^n by the whole number, then we must multiply the den^r by this divisor. By this method $\frac{3}{5} \div 4$ would $= \frac{3}{5 \times 4} = \frac{3}{20}$.

To prove this, observe in the fractions $\frac{3}{5}$ and $\frac{3}{20}$, that in the first fr^n the unit is divided into 5 equal parts, (see

Fig. 3,) and therefore each = $\frac{1}{5}$; and, in the second, the same unit is divided into 20 equal parts, and therefore each = $\frac{1}{20}$:

FIG. 3.



now it is plain, from Fig. 3, that *one*-fifth, when divided by 4, becomes *one*-twentieth; so *three*-fifths divided by 4 become *three*-twentieths;

$$\therefore\ \frac{3}{5} \div 4 = \frac{3}{20},\ \text{which} = \frac{3}{5 \times 4};\ \text{hence,}$$

We multiply the den^{r} by the given divisor, and leave the num^{r} unaltered.

The following are specimens of the mode of working.

$$\frac{9}{10} \div 3 = \frac{9 \div 3}{10} = \frac{3}{10}.$$

$$\frac{9}{10} \div 6 = \frac{9}{10 \times 6} = \frac{9}{60}.$$

OBS. Since it rarely happens that we can divide any num^{r} or den^{r} by any chance number, we therefore, in multiplying a fr^{n}, have, as a general rule, to multiply the num^{r}; and in dividing a fr^{n}, to multiply the den^{r}.

Exs. 4.

III. **Divide $\frac{8}{9}$ by 2, 3, 4, 5, 6, 7, 8, 9, successively.**
IV. **Divide $\frac{36}{49}$ by 2, 3, 4, 7, 8, 9, 18, successively.**

31. We have now shown that to multiply a fr^{n} by a whole number, we multiply the *num^{r}*, and to divide a fr^{n} we multiply the *den^{r}*; therefore if we multiply both num^{r} and den^{r} by the same quan^{y}, we shall have both multiplied and divided the fr^{n} by the same number; but since to multiply a quan^{y} by any number, and then to divide by the same, leaves

the original quan^y unaltered, therefore to multiply both num^r and den^r of a fr^n will not alter its value:

$$\text{Thus, } \frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20};$$

and by observing Fig. 3, we shall see that since 3 of the large divisions, or *fifths*, = 12 of the small divisions, or *twentieths*, we have by inspection the same result, viz. $\frac{3}{5} = \frac{12}{20}$.

32. So also, since by dividing the num^r we *divide* the whole fr^n, and by dividing the den^r we *multiply* the same fr^n; therefore, if we divide both num^r and den^r by the same quantity, the fr^n will remain unaltered in value:

$$\text{Thus, } \frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5};$$

by a process opposite to that in the last Art.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

33. We have shown that both num^r and den^r of a fr^n can be divided by the same number without altering the value of the fr^n; and if the num^r and den^r cannot be exactly divided by any whole number, the fr^n is said to be in its lowest terms, because it is expressed in the lowest possible numbers.

Thus, $\frac{3}{4}$, $\frac{5}{6}$ are fr^{ns} in their lowest terms, because 3 and 4 in the first fr^n, and 5 and 6 in the second fr^n, have no common divisor, or C. M.

But $\frac{8}{12}$ is not in its lowest terms, for both num^r and den^r can be divided by 4; and the fr^n will then be $\frac{2}{3}$. So also, in fr^n $\frac{15}{25}$, 5 is G. C. M. of num^r and den^r; and the fr^n in its lowest terms = $\frac{3}{5}$.

It is plain, therefore, that if any fr^n be not in its lowest terms, and we wish so to reduce it, both num^r and den^r must be divided by the greatest possible divisor, *i. e.* by their G. C. M.

In working a sum of this kind we should express the operation thus:

$$\frac{15}{25}=\frac{15\div5}{25\div5}=\frac{3}{5},$$

but when, in future, it is necessary to use this operation, it may be thus written, $\frac{15}{25}=\frac{3}{5}$, the intermediate step being omitted.

34. When the num^r and den^r of a fr^n are both small numbers, the G. C. M. may be often found by *inspection*, or sometimes *some* C. M. may be found, though not the greatest; and, by dividing by it, the fr^n may *partly* be reduced; and then a C. M. of the fr^n so reduced may be found, and a second division performed, till we reduce the fr^n, if not to the lowest, at any rate to lower terms.

Thus, $\frac{35}{245}=\frac{7}{49}=\frac{1}{7}$ is now reduced to lowest terms.

It is worth notice, that every number ending in 5 or 0 must have 5 for a divisor, and every even number can be divided by 2.

But if mere inspection will not tell us the G. C. M. of num^r and den^r, it must be found by the method already given (14) for finding the G. C. M. of two given numbers.

Exs. 5. Reduce to their lowest terms

1. $\frac{24}{40}$	3. $\frac{275}{425}$	5. $\frac{1197}{2471}$	7. $\frac{935}{1573}$	9. $\frac{459}{1530}$	11. $\frac{343}{539}$
2. $\frac{85}{165}$	4. $\frac{1368}{3096}$	6. $\frac{2241}{2700}$	8. $\frac{5705}{6559}$	10. $\frac{1111}{9090}$	12. $\frac{891}{945}$

TO REDUCE AN IMPROPER FRACTION TO A WHOLE OR MIXED NUMBER.

35. Ex. To reduce $\frac{17}{5}$ to a whole or mixed number. We have before shown (22) that this fr^n expresses that 17 is to

be divided by 5; now, if the 17 were exactly divisible by 5, the quotient would be a whole number; but if not, as in this case, then since 15 is the largest multiple of 5 below 17, we can divide the 15 by 5, and the quotient is 3; but the division of the remaining 2 by the 5 cannot be *performed:* it must therefore be *understood,* by placing the divisor 5 underneath the 2 as a den^{r}, and thus $\frac{2}{5}$ may be said to be the quot^{t} of 2 when divided by 5: the whole quotient will, therefore, be 3 and $\frac{2}{5}$; and the operation may be thus written—

$$\frac{17}{5}=\frac{15+2}{5}=\frac{15}{5}+\frac{2}{5}=3+\frac{2}{5},$$

or, as it is generally written, $= 3\frac{2}{5}$.

Also, since 5-fifths = 1, therefore 15-fifths = 3, and 17-fifths must = 3 and 2-fifths, or $3\frac{2}{5}$, as just shown.

36. In writing the whole operation as fully as has been done above, it is necessary to find what is the rem^{r} after the num^{r} has been divided by the den^{r}. It must then be placed as the rem^{r} 2 has been placed in the above Ex., and the 15 is of course found by subtracting this rem^{r} from the whole num^{r}. Where the num^{r} and den^{r} are small, this rem^{r} may be found without working any div^{n} sum on the paper; but if the numbers are large, a div^{n} sum must be worked before the above operation can be shown.

Thus: Ex. 2. Reduce $\frac{327}{19}$ to a whole or mixed number.

19) 327 ($17\frac{4}{19}$
19
137
133
4
19

Here, the rem^{r} being 4, the largest multiple of 19 below 327 is 323, which contains 19, 17 times, therefore we have $\frac{327}{19}=\frac{323+4}{19}=\frac{323}{19}+\frac{4}{19}=17\frac{4}{19}$;

or, as usually written, $\frac{327}{19}=17\frac{4}{19}$.

Exs. 6. Reduce to whole or mixed numbers

1. $\frac{24}{7}$ 3. $\frac{288}{15}$ 5. $\frac{12211}{113}$ 7. $\frac{10001}{99}$ 9. $\frac{1759}{176}$ 11. $\frac{1749}{55}$
2. $\frac{82}{11}$ 4. $\frac{603}{28}$ 6. $\frac{5576}{17}$ 8. $\frac{9999}{100}$ 10. $\frac{8481}{844}$ 12. $\frac{4001}{87}$

TO REDUCE A MIXED NUMBER TO AN IMPROPER FRACTION.

37. Ex. To reduce $3\frac{2}{5}$ to an improper fr^{n}. That we may explain this operation properly, we must observe, that in order to express two or more numbers in one sum, these must all be of one kind or denomination; thus, in a simple addition sum, we add units to units, tens to tens, &c.; and in a compound addition sum we add together like quantities, as pounds to pounds, shillings to shillings, &c.

If, therefore, I have to express in one sum 3 units and 2 fifths, these two quantities must be reduced to the same kind. Now, I cannot reduce the fifths to a whole number, because two-fifths are less than unity; I must, therefore, reduce the 3 units to fifths, that is, I must express 3 as a fr^{n} with a den^{r} 5.

Thus, by (25) $3=\frac{3\times 5}{5}=\frac{15}{5}$; adding then the 15-fifths to the two-fifths, I have the sum 17-fifths, or $\frac{17}{5}$.

The whole operation may be thus expressed:

$$3\frac{2}{5}=\frac{3\times 5+2}{5}=\frac{15+2}{5}=\frac{17}{5}.$$

38. It will be seen that this Ex. is exactly the reverse of that in (35); and we may observe, that the num^{r} of the improper fr^{n}, viz. 17, is obtained by multiplying the whole

number 3 by the denʳ 5, and adding the former numʳ 2 : thus, in working the above Ex., I should say 5 times 3 are 15 and 2 are 17.

When, then, we have in future Exˢ to reduce a mixed number, $3\frac{2}{5}$, to an improper frⁿ, we should merely write $3\frac{2}{5} = \frac{17}{5}$, where all the intermediate work can be performed mentally, when the numbers are not large. We could thus write $9\frac{7}{12} = \frac{115}{12}$, because 12 times 9 are 108 and 7 are 115.

Exs. 7. Reduce to improper fractions

1. $1\frac{6}{7}$
2. $4\frac{1}{8}$
3. $13\frac{2}{15}$
4. $276\frac{5}{29}$
5. $130\frac{43}{114}$
6. $100\frac{161}{193}$
7. $175\frac{1}{175}$
8. $45\frac{121}{343}$
9. $145\frac{10}{111}$
10. $77\frac{7}{90}$

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR.

39. It has been shown (37) that numbers, whether whole or fractional, cannot be added together, without being reduced to the same denomⁿ or kind.

We must, therefore, show how to reduce fractions to a common denʳ; and then they will be of the same denomination.

$$\text{Ex.} \quad \frac{3}{4}, \ \frac{5}{6}, \ \frac{7}{8}, \ \frac{9}{10}.$$

Now, in order that the fractions, when reduced, may still consist of as small numbers as possible, we shall always find their *least* common denʳ.

Also, since we can alter the form of fractions only by multiplying or dividing both numʳ and denʳ, we must therefore multiply these present denʳˢ, each one by some number, so that they all shall be changed into the new denʳ; and for this purpose the new denʳ must be divisible by all these denʳˢ, and therefore must be their L. C. M.

Working according to (12) we find the L. C. M. of 4, 6, 8, 10, to be 120. We now choose such multipliers as shall make 4, 6, 8, 10, become 120: these will be found by dividing 120 by the 4, 6, 8, 10; and are 30, 20, 15, 12.

Now, we know (31) that if we multiply the den^r of a fr^n by any number, we must also multiply the num^r by the same number, or else the fr^n will be altered in value. We therefore work thus:

$$\frac{3}{4}=\frac{3\times 30}{4\times 30}=\frac{90}{120}$$
$$\frac{5}{6}=\frac{5\times 20}{6\times 20}=\frac{100}{120}$$
$$\frac{7}{8}=\frac{7\times 15}{8\times 15}=\frac{105}{120}$$
$$\frac{9}{10}=\frac{9\times 12}{10\times 12}=\frac{108}{120}$$

and these fractions may be thus written, $\frac{90,\ 100,\ 105,\ 108}{120}$ showing at once that they have the same den^r.

40. In working sums of this kind, we should have to write down the process of finding the L. C. M. of all the den^{rs}, unless we could see it by *inspection*, which cannot be often done, when the numbers are large.

If it can be done, we must then merely write as the first line of the work L. C. M. $=120$, or any number that it may chance to be.

Exs. 8. Reduce to their least common denominator

1.	$\frac{13}{18}, \frac{4}{5}, \frac{7}{9}, \frac{11}{40}, \frac{15}{16}$	4.	$\frac{3}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$	7.	$\frac{7}{15}, \frac{27}{35}, \frac{5}{72}, \frac{51}{63}, \frac{29}{54}$
2.	$\frac{1}{7}, \frac{3}{11}, \frac{5}{21}, \frac{9}{56}, \frac{3}{14}$	5.	$\frac{2}{3}, \frac{4}{5}, \frac{7}{8}, \frac{11}{12}, \frac{83}{120}$	8.	$\frac{14}{17}, \frac{1}{11}, \frac{11}{15}, \frac{17}{33}$
3.	$\frac{11}{15}, \frac{13}{16}, \frac{14}{17}, \frac{17}{18}$	6.	$\frac{3}{4}, \frac{13}{20}, \frac{11}{18}, \frac{27}{70}, \frac{41}{90}$	9.	$\frac{11}{12}, \frac{59}{60}, \frac{17}{24}, \frac{1}{120}$

TO ADD TOGETHER ANY NUMBER OF FRACTIONS.

41. These may be all proper fractions, or some of them may be proper, some improper, and some mixed numbers.

We have already seen that in order to add fractions together, we must reduce them to a com. den^r. Now, this has been done in the last two Art^s; and therefore we have only to conclude the operation in the last Ex. by performing the operation of adding the fractions so prepared. Now we observe, that whatever number of quantities of any kind are to be added together, the *kind* or denomination is not thereby changed: thus, 7 *months* added to 5 *months* would be 12 *months;* so, 4-*twelfths* + 3-*twelfths* = 7-*twelfths*, which, expressed in figures, gives $\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$; hence it is plain that we have only to add the num^{rs} together, and retain the *same* den^r; therefore, so doing in the last Ex., we have

$$\text{sum of the fr}^{ns}=\frac{90+100+105+108}{120}=\frac{403}{120}$$
$$=3\tfrac{43}{120}\text{ by (35).}$$

If the proper fr^n $\frac{43}{120}$ had not been in its lowest terms, it must have been reduced to that state before the sum could be called complete.

42. If some of the fr^{ns} to be added together be mixed numbers, we must first add the frac^l parts precisely as above; and to this sum then add the amount of the whole numbers. Thus, if the Ex. which we have just worked had been as follows, $3\frac{3}{4}+\frac{5}{6}+6\frac{7}{8}+15\frac{9}{10}$ (where the insertion of the whole numbers 3, 6, and 15, is the only alteration), we should then have, as before,

$$\text{sum of the fractional parts}=3\tfrac{43}{120};$$

therefore, adding in the whole numbers, we have the whole sum $=3+6+15+3\frac{43}{120}=27\frac{43}{120}$.

We will now work a second example completely.

Find the sum of $5\frac{2}{3}+1\frac{3}{4}+\frac{4}{7}+4\frac{5}{6}+\frac{7}{8}$.

$$\overline{3},\ \overline{4},\ 7,\ 6,\ 8,$$
$$7,\ \overline{2}\times3,\ \ 2\times2\times2,$$
$$\text{L. C. M.}=7\times3\times8=21\times8=168.$$

$$\frac{2}{3}=\frac{2\times 56}{3\times 56}=\frac{112}{168}$$
$$\frac{3}{4}=\frac{3\times 42}{4\times 42}=\frac{126}{168}$$
$$\frac{4}{7}=\frac{4\times 24}{7\times 24}=\frac{96}{168}$$
$$\frac{5}{6}=\frac{5\times 28}{6\times 28}=\frac{140}{168}$$
$$\frac{7}{8}=\frac{7\times 21}{8\times 21}=\frac{147}{168}$$

therefore, sum of frac^l parts $=\dfrac{112+126+96+140+147}{168}$

$=\dfrac{621}{168}=3\frac{117}{168}=3\frac{39}{56}$ [in lowest terms]

and whole sum $=5+1+4+3\frac{39}{56}=13\frac{39}{56}$.

Obs. If there be any improper fractions in the proposed example, they may be reduced to mixed numbers before we begin to reduce to L. C. D.; or, if they are not very large, we may proceed with them as with the proper fractions.

Exs. 9. Find the value of

1. $\frac{3}{4}+\frac{4}{5}+\frac{5}{12}+\frac{11}{16}$
2. $\frac{7}{9}+\frac{5}{8}+\frac{11}{30}+\frac{14}{27}$
3. $\frac{5}{9}+\frac{7}{8}+\frac{11}{16}+\frac{13}{20}+\frac{1}{2}$
4. $\frac{2}{5}+\frac{17}{15}+\frac{4}{25}+\frac{7}{27}$
5. $10\frac{3}{8}+1\frac{5}{12}+\frac{7}{10}+4\frac{13}{15}$
6. $17+1\frac{3}{14}+\frac{5}{7}+13\frac{15}{28}$
7. $5\frac{7}{11}+3\frac{8}{9}+\frac{15}{22}+17\frac{8}{35}+1\frac{7}{10}$
8. $7\frac{1}{12}+8\frac{1}{13}+9\frac{1}{14}+10\frac{1}{15}$

SUBTRACTION.

43. The examples under this head will be of different kinds.

1st. When there are two fr^{ns}, both proper, as $\frac{3}{10}-\frac{1}{4}$, I.

2nd. When there are two fr^{ns}, one or both of which are mixed numbers, as $7\frac{3}{4}-3\frac{2}{5}$, II.; $8\frac{2}{5}-2\frac{3}{7}$, III.; $8-2\frac{5}{7}$, IV.

If there be any examples containing improper fr^{ns}, such examples may either be worked as I.; or, by reducing the improper fr^{ns} to mixed numbers, we bring them under II., III., or IV.

The difference between II., III., and IV. consists in this, that

in II. the former frac^l part $\frac{3}{4}$ is > the latter $\frac{3}{5}$;
in III. „ „ „ $\frac{2}{5}$ is < „ $\frac{3}{7}$; (A)
in IV. „ „ „ is 0, and ∴ < $\frac{5}{7}$.

44. Under the head of Subtraction will also come examples in which there are more than two fractions; and wherein we have to find the difference between the sum of one set of fractions, and the sum of another set, as in the following

Ex. $5\frac{2}{3} + 4\frac{3}{8} - \frac{1}{2} + 3\frac{5}{6} - 7\frac{3}{7}$;

wherein the signs + and – show that I am to take both $\frac{1}{2}$ and $7\frac{3}{7}$, that is, the sum of $\frac{1}{2}$ and $7\frac{3}{7}$, from the sum of $5\frac{2}{3} + 4\frac{3}{8} + 3\frac{5}{6}$.

Def. If any numbers, either whole or frac^l, are joined together by one or more of the signs +, –, &c., the whole set of numbers thus connected is often termed an *expression*.

45. Just as it has been shown that fractions cannot be added together until they are reduced to a C. D., so it will appear that one fr^n cannot be subtracted from another till the two fr^{ns} are reduced to a C. D. Thus, if I had to take 3 pence from 1 shilling, I must bring the 1*s.* to pence, and then take 3*d.* from 12*d.*, obtaining a rem^r 9*d.*

Taking, then, Ex. I. $\frac{3}{10} - \frac{1}{4}$, we may see, by *inspection*, that L.C.D. = 20;

$$\text{therefore } \frac{3}{10} = \frac{3 \times 2}{10 \times 2} = \frac{6}{20}; \quad \frac{1}{4} = \frac{1 \times 5}{4 \times 5} = \frac{5}{20};$$

$$\text{and} \qquad \frac{3}{10} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20}. \qquad \text{(B)}$$

When the L. C. D. is not large, say under 150, a pupil will, with a little experience, soon be able at once to write down the line (B), without working the two previous operations on paper.

46. It was said in line (A) $\frac{2}{5} < \frac{3}{7}$; a beginner cannot see this at once; but he may do so very readily—thus: we already know that no two quantities can be compared, to see which is the larger, unless they be of the same kind; so, two fr^{ns} cannot be compared unless reduced to a C. D. *Any* C. D. will, however, do for the purpose of merely telling which of the two fr^{ns} is the larger; now the product of the two den^{rs} is always *one* C. D., but not always the *least;* therefore since 35 is a C. D. of $\frac{2}{5}$ and $\frac{3}{7}$,

$$\frac{2}{5}-\frac{3}{7}=\frac{2\times7-5\times3}{5\times7}=\frac{14-15}{35}; \quad \text{(C)}$$

$$\text{and since } 14<15, \text{ therefore } \frac{14}{35}<\frac{15}{35}, \text{ or } \frac{2}{5}<\frac{3}{7}.$$

Now this result may be very rapidly obtained without any work on paper; for if we observe line (C), we notice that the numbers 14 and 15, which numbers alone I wish to compare, are found by multiplying the num^{r} of the 1st fr^{n} by the den^{r} of the 2nd, and the den^{r} of the 1st by the num^{r} of the 2nd. Thus, beginning with the num^{r} of the 1st, I say, *mentally*, $2 \times 7 = 14$: and again, $3 \times 5 = 15$; therefore, since the first product is less than the second, so also the first fr^{n} is less than the second.

Taking one or two more Exs. of this process—

1. To compare $\frac{3}{7}$ and $\frac{2}{9}$: I say, $3 \times 9 = 27$; $7 \times 2 = 14$; therefore the former fr^{n} is the larger.

2. To compare $\frac{3}{8}$ and $\frac{7}{12}$: I say, $3 \times 12 = 36$; $8 \times 7 = 56$; therefore the latter fr^{n} is the larger.

It will be noticed that the C. D. to which we have just now mentally reduced $\frac{3}{8}$ and $\frac{7}{12}$ is 96; but when we come to the *actual sub^{n}*, we shall not use *this* C. D., but the L. C. D. 24

47. In Exs. II., III., IV., we may bring the mixed numbers into improper fractions, and work them as in Ex. I.; but since, when the numbers are large, the work becomes heavy, it is better generally to employ the same process that we use in Comp^{d} Sub^{n}. Take as patterns the three following:

A		B		C	
s.	*d.*	*s.*	*d.*	*s.*	*d.*
18	9	18	2	18	0
15	3	15	5	15	5
3	6	2	9	2	7

Now, in these Exs. it will be seen, that where the number of pence in the upper line exceeds that in the lower line, as in (A), we can at once subtract the 3*d.* from the 9*d.*, and the 15*s.* from the 18*s.*; giving a complete rem^{r} 3*s.* 6*d.* So in Ex. II., $7\frac{3}{4} - 3\frac{2}{5}$, since $\frac{3}{4} > \frac{2}{5}$, therefore I can take the $\frac{2}{5}$ from the $\frac{3}{4}$, and the 3 from the 7.

The difference of the frac^{l} parts $= \frac{3}{4} - \frac{2}{5} =$ (when reduced to L. C. D.) $\frac{15-8}{20} = \frac{7}{20}$; and diff^{e} of the whole numbers $= 7 - 3 = 4$; therefore, difference of the complete fractions

$$= 4 + \tfrac{7}{20},$$
$$= 4\tfrac{7}{20}.$$

The following Ex. is written as it should be worked.

Find the value of $9\frac{3}{8} - 5\frac{1}{5}$.

$$\text{L. C. D.} = 40.$$
$$\therefore \frac{3}{8} - \frac{1}{5} = \frac{15-8}{40} = \frac{7}{40}.$$
$$\text{And } 9 - 5 = 4.$$
$$\therefore 9\tfrac{3}{8} - 5\tfrac{1}{5} = 4\tfrac{7}{40}.$$

48. But in (B) and (C) the number of pence in the lower line is larger than in the upper, and in both cases I must borrow 1*s.* or 12*d.* from the 18*s.* in the upper line, and

add this 12*d.* to the pence in the upper line, if there be any; and when I have sub^d the 5*d.* from the 14*d.* in (B), leaving rem^r 9*d.*, and from 12*d.* in (C), leaving rem^r 7*d.*, I subtract 15*s.* from 17*s.*, (not 18*s.*, because I have just borrowed 1*s.* from the 18*s.*) and in both cases I have a rem^r 2*s.*; therefore the whole rem^r in (B) is 2*s.* 9*d.*, and in (C) is 2*s.* 7*d.*

I now apply this process to Exs. III. and IV., viz. $8\frac{2}{5}-2\frac{3}{7}$, III.; and $8-2\frac{5}{7}$, IV. As, then, in the Comp^d Sub^n I borrowed 1*s.* from the higher den^n 18*s.*, and added it to the pence in the upper row, if there were any, so I borrow 1 from the 8, and add it to the former fr^n, if there be any; thus in III. I add the 1 to the $\frac{2}{5}$, and then subtracting $\frac{3}{7}$, I say—

$$1\frac{2}{5}-\frac{3}{7}=\frac{7}{5}-\frac{3}{7}=\frac{49-15}{35}=\frac{34}{35},$$

then, taking the 8 as 7, I have $7-2=5$; therefore, combining the results of the two subtractions, I have

$$8\tfrac{2}{5}-2\tfrac{3}{7}=5+\tfrac{34}{35}=5\tfrac{34}{35}.$$

If, however, it is desired to have the result in the form of an improper fraction, as will be found most convenient in Exs. which involve Mult^n, we may bring the two quantities to improper fractions, as mentioned in (47); thus

$$8\tfrac{2}{5}-2\tfrac{3}{7}=\frac{42}{5}-\frac{17}{7}=\frac{294-85}{35}=\frac{209}{35}.$$

The following Ex. is written as it should be worked.

Find the value of $17\frac{1}{2}-11\frac{5}{6}$.

$\frac{1}{2}<\frac{5}{6}$ ∴ borrowing 1, I have $1\frac{1}{2}-\frac{5}{6}=\frac{3}{2}-\frac{5}{6}=\frac{9-5}{6}=\frac{4}{6}=\frac{2}{3}$.

And $16-11=5$.

∴ $17\frac{1}{2}-11\frac{5}{6}=5\frac{2}{3}$.

49. In Ex. IV., $8-2\frac{5}{7}$, I borrow 1 from the 8, but as I have no former fr^n to which I may add this 1, as in Ex. III., therefore subtracting the second fr^n $\frac{5}{7}$ from the 1, I have

$$1-\frac{5}{7}=\text{(when red}^{\text{d}}\text{ to L. C. D.)}\ \frac{7}{7}-\frac{5}{7}=\frac{7-5}{7}=\frac{2}{7};\ \text{(D)}$$

and taking 7 instead of 8, as before, I have $7-2=5$; therefore, I should work as follows—

$$1-\frac{5}{7}=\frac{7}{7}-\frac{5}{7}=\frac{2}{7}.$$

And $7-2=5$.

$\therefore 8-2\frac{5}{7}=5\frac{2}{7}$.

50. In the line (D) I may observe that when the number 1 is expressed as a fr^{n}, its num^{r} is made the same as the den^{r} of the fr^{n} which I have to subtract: thus, 7 is the new num^{r}, and is also the den^{r} of the original fr^{n}; whether, therefore, I subtract the smaller num^{r} from the new num^{r} or from the den^{r} of the smaller fr^{n} I obtain the same result, viz. a rem^{r} 2; and I might say at once $1-\frac{5}{7}=\frac{2}{7}$, where the 2 has been found by subtracting the num^{r} 5 from its own den^{r} 7. Working thus, I should have written the whole Ex. as follows:

$$8-2\tfrac{5}{7}=(7-2)+(1-\tfrac{5}{7})=5\tfrac{2}{7},$$

only that in common use I should not write the figures enclosed in the two parentheses.

51. Lastly: when there are more than two fractions, as in

Ex. v. $5\frac{2}{9}+4\frac{3}{8}-\frac{1}{2}+3\frac{5}{6}-7\frac{3}{7}$.

Here, omitting for the present the whole numbers, I have to subtract the sum of $\frac{1}{2}$ and $\frac{3}{7}$ from the sum of $\frac{2}{9}+\frac{3}{8}+\frac{5}{6}$. Since the two fr^{ns} to be subtracted are together less than the three from which I have to subtract them, no difficulty will occur, and I proceed as in an Ex. in Addition, taking care to arrange the fr^{ns} to be subtracted last:

$$\frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7};$$

$$9,\ 8,\ 6,\ \overline{2},\ 7.$$

$$3\times 3,\quad 2\times 2\times 2,\quad \overline{2}\times\overline{3},\quad 7.$$

$$\text{L. C. D.} = 9\times 8\times 7 = 72\times 7 = 504.$$

$$\frac{2}{9}=\frac{2\times 56}{9\times 56}=\frac{112}{504}\qquad \text{See Art. (9).}$$

$$\frac{3}{8}=\frac{3\times 63}{8\times 63}=\frac{189}{504}$$

$$\frac{5}{6}=\frac{5\times 84}{6\times 84}=\frac{420}{504}$$

$$\frac{1}{2}=\frac{1\times 252}{2\times 252}=\frac{252}{504}$$

$$\frac{3}{7}=\frac{3\times 72}{7\times 72}=\frac{216}{504}$$

$$\therefore \frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7}=\frac{112+189+420-252-216}{504}$$

$$=\frac{721-468}{504}=\frac{253}{504}$$

therefore, bringing in the whole numbers with their proper signs,

$$\text{the whole expression} = 5+4+3-7+\frac{253}{504}=12-7+\frac{253}{504}$$

$$=5\tfrac{253}{504}.$$

52. We will take one more example of this kind.

Ex. VI. $3\tfrac{4}{5}-4\tfrac{1}{2}+7\tfrac{2}{7}-\tfrac{3}{4}.$

Now here $\tfrac{1}{2}+\tfrac{3}{4}$ are $>\tfrac{4}{5}+\tfrac{2}{7}$; but since this cannot be seen by mere *inspection*, I proceed as in Ex. v., as though I did not know this, and provide for the difficulty at the proper time. Placing the frns to be subtracted last, I have

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}$$

$$5,\ 7,\ \overline{2},\ 4,$$

$$\text{L. C. D.} = 5\times 7\times 4 = 140.$$

$$\frac{4}{5}=\frac{4\times 28}{5\times 28}=\frac{112}{140}$$

$$\frac{2}{7}=\frac{2\times 20}{7\times 20}=\frac{40}{140}$$

$$\frac{1}{2}=\frac{1\times 70}{2\times 70}=\frac{70}{140}$$

$$\frac{3}{4}=\frac{3\times 35}{4\times 35}=\frac{105}{140}$$

$$\text{therefore } \frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$$

$$=\frac{152-175}{140}. \qquad \text{(E)}$$

Since, then, 175 > 152, it now appears that I ought to have borrowed 1; but rather than begin the work again, I proceed thus: from the $\frac{152}{140}$ in line (E) I take as much of the $\frac{175}{140}$ as I can, viz. $\frac{152}{140}$, and there will remain $\frac{23}{140}$, which cannot be subtracted, and before which I place the sign (−), to shew that it has yet to be subtracted; but if I now borrow 1, I can subtract this $\frac{23}{140}$ from the 1, and remember to count the first (+) whole number in Ex. VI. (*i. e.* the 3) as one less than it now stands. The whole operation may be thus written, recommencing with the line before (E):

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$$

$$=\frac{152-175}{140}=\frac{-23}{140}.$$

Therefore borrowing the 1, $1-\frac{23}{140}=\frac{140}{140}-\frac{23}{140}=\frac{117}{140}$,

and writing 2 for 3 in Ex. VI., and inserting the whole numbers with their proper signs,

the whole expression $=2+7-4+\frac{117}{140}=9-4+\frac{117}{140}$

$=5\frac{117}{140}$.

Exs. 10. Exhibit in one term each of the following expressions:

1. $\frac{5}{7}-\frac{2}{3}$
2. $\frac{22}{45}-\frac{5}{18}$
3. $\frac{29}{56}-\frac{5}{14}$
4. $\frac{15}{16}-\frac{11}{48}$
5. $\frac{25}{36}-\frac{13}{64}$
6. $\frac{18}{19}-\frac{17}{18}$
7. $5\frac{3}{8}-4\frac{2}{7}$
8. $17\frac{9}{10}-11\frac{5}{12}$
9. $8\frac{1}{2}-4\frac{3}{5}$
10. $7\frac{3}{4}-3\frac{11}{12}$
11. $8-3\frac{7}{9}$
12. $17-\frac{1}{2}$
13. $38-27\frac{11}{16}$
14. $3\frac{2}{5}+4\frac{3}{4}-7\frac{1}{2}$
15. $18\frac{1}{4}-7\frac{11}{12}+\frac{5}{9}$
16. $7\frac{1}{2}+4\frac{5}{6}-\frac{7}{12}+3\frac{3}{10}-5\frac{11}{16}$
17. $8\frac{4}{5}-6\frac{3}{7}-\frac{8}{9}+15\frac{1}{3}-3\frac{4}{27}$
18. $18\frac{1}{3}-5\frac{2}{7}+4\frac{8}{15}-\frac{2}{9}-\frac{5}{12}$

MULTIPLICATION.

53. We will show, as we proceed, that the operation of multiplication in fractions may be expressed in two ways, and that the word *of* and the sign (×) placed between fr^{ns}, have the same meaning: thus, we have to prove that

$$\frac{5}{7} \text{ of } \frac{2}{3} \text{ of } 3\frac{3}{8} \text{ and } \frac{5}{7} \times \frac{2}{3} \times 3\frac{3}{8} \text{ have the same value.} \qquad \text{(F)}$$

The pupil must here clearly call to mind the method of multiplying and dividing fractions by whole numbers. (31)

I will first begin with a pair of fractions, as $\frac{5}{7} \times \frac{2}{3}$.

It has been proved that $\frac{2}{3}$ means *one-third* of 2; therefore I have now to multiply $\frac{5}{7}$ by one-third of 2: hence the product ought to be one-third of the product found by multiplying by 2 alone: *i. e.* if I multiply by 2, and then take one-third of that product, I shall obtain the correct result.

Multiplying $\frac{5}{7}$ by 2, I have $\frac{5}{7} \times 2 = \frac{5 \times 2}{7}$;

and since to take one-third of any quantity is to divide it by 3, I divide this last fr^{n} by 3, and have

$$\frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7} \div 3 = \frac{5 \times 2}{7 \times 3}. \qquad \text{(G)}$$

Commencing now with the expression $\frac{5}{7}$ of $\frac{2}{3}$: we know that $\frac{5}{7}$ means 5 times one-seventh; therefore, if I have to take five-sevenths of any quantity, I first take *one*-seventh of it, and then multiply the quot^{t} so obtained, by 5: *i. e.* to take $\frac{5}{7}$ of $\frac{2}{3}$, I must divide the $\frac{2}{3}$ by 7, and multiply it by 5: hence, according to our rules of mult^{n} and div^{n} of fractions by whole numbers, I multiply the den^{r} of the $\frac{2}{3}$ by 7, and the num^{r} by 5, and have

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{5 \times 2}{7 \times 3}. \qquad \text{(H)}$$

Comparing (G) and (H), I now perceive that

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{5}{7} \times \frac{2}{3}.$$

54. The following explanation may, perhaps, be intelligible to some pupils:

In multiplying a quantity, whether whole or fractional, by a whole number, I repeat that quantity as many times as there are units in the multiplier:

Thus, 7 × 5, or 7 times 5, means that I am to repeat the 5 as many times as there are units in 7;

So, $7 \times \frac{5}{7}$ shows that I am to repeat $\frac{5}{7}$ as often as there are units in 7;

Also, $1 \times \frac{5}{7}$ shows that I am to repeat $\frac{5}{7}$ as often as there are units in 1, *i. e.* 1 time, or once:

Therefore, $\frac{2}{3} \times \frac{5}{7}$ ought to mean, that I am to repeat $\frac{5}{7}$ as often as there are units in $\frac{2}{3}$.

Now, $\frac{2}{3}$ does not contain 1 unit,—it only contains $\frac{2}{3}$ of 1 unit; therefore, $\frac{2}{3} \times \frac{5}{7}$ is $\frac{5}{7}$ repeated $\frac{2}{3}$ of once, *i. e.* it $= \frac{2}{3}$ of once $\frac{5}{7}$, or $\frac{2}{3}$ of $\frac{5}{7}$; and this is the required result, as in last article.

55. From what has been shown in (53), we may see that

$$\text{since} \quad \frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7 \times 3} = \frac{10}{21},$$

$$\text{and} \quad \frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21};$$

$$\text{therefore} \quad \frac{5}{7} \times \frac{2}{3} = \frac{2}{3} \times \frac{5}{7}:$$

and since the sign (×) and the word *of* between fr^{ns} have been shown to have the same meaning, we see that

$$\frac{5}{7} \text{ of } \frac{2}{3} = \frac{2}{3} \text{ of } \frac{5}{7}.$$

56. Proceeding with the expression (F) in (53), and changing $3\frac{3}{5}$ into an imp^r fr^n, we have

$$\frac{5}{7} \text{ of } \frac{2}{3} \text{ of } 3\frac{3}{5} = \frac{5 \times 2}{7 \times 3} \text{ of } \frac{18}{5};$$

and since $\frac{5 \times 2}{7 \times 3}$ is one fr^n, and $\frac{18}{5}$ another fr^n, I may use the method of multiplying, which has been shown to be true for *two* fractions;

$$\text{therefore } \frac{5}{7} \text{ of } \frac{2}{3} \text{ of } 3\tfrac{3}{5} = \frac{5\times2\times18}{7\times3\times5}: \qquad \text{(I)}$$

and if the expression in (F) had consisted of four fr^{ns} connected by *of* or (×), the first three might have been reduced to one fr^{n}, as in (I), and we should then have had but *two* fr^{ns}, which we have just shown how to reduce to one. Therefore the results proved in lines (F), (G), (H), (I), are true, whatever number of fr^{ns} may be contained in any compound fraction.

57. To return to (I). It has been proved that if a num^{r} and den^{r} have any common measure, both of them may be divided by this common measure; therefore if in the right-hand side of equation (I), we find any factors common to both num^{r} and den^{r}, these should be struck out before we complete the multiplication.

We see in this case that there is a factor 5 in the num^{r} and den^{r}; also, that 18 in the num^{r} and 3 in the den^{r} have a common divisor 3: dividing by these common factors,

$$\text{the expression stands thus, } \frac{\overset{1}{\cancel{5}}\times2\times\overset{6}{\cancel{18}}}{7\times\underset{1}{\cancel{3}}\times\underset{1}{\cancel{5}}}.$$

The remaining factors in the num^{r} are 1, 2, 6; and in the den^{r} are 7, 1, 1; these form a fraction $\frac{2\times6}{7}=\frac{12}{7}=1\tfrac{5}{7}$. This process of striking out factors from num^{r} and den^{r} is called *cancelling*, and is to be continued as long as any factors can be found common to both num^{r} and den^{r}. When the numbers are large, this cancelling cannot be readily performed, except by those who have had some experience.

OBS. A factor 1 left in a num^{r} or den^{r} will not alter the value of the other factors which remain, when we multiply together all those in the num^r, and all those in the den^r; and hence the figures 1 need not be written, as in the above Ex.; but it must be noticed, that if no other factors remain, the value of the num^r or den^r will consist of the product of as many figures 1 as there were factors in the num^r or den^r before cancelling; *i. e.* it will be = 1. If, therefore, all the factors in any num^r or den^r have been cancelled, the num^r or den^r of the fraction after cancelling will be 1. But when this occurs in the den^r, it need not be expressed, but the result written as a whole number.

Ex. II. $$8\tfrac{2}{5} \times 35 \times \tfrac{17}{24} = \frac{\overset{7}{\cancel{42}}}{\cancel{5}} \times \frac{\overset{7}{\cancel{35}}}{1} \times \frac{17}{\underset{4}{\cancel{24}}}$$

$$= \frac{7 \times 7 \times 17}{4} = \frac{833}{4} \qquad \text{(K)}$$

$$= 208\tfrac{1}{4}.$$

Here 42 and 24 had a C. M. 6, and 5 and 35 had a C. M. 5.

The fr^n $\frac{833}{4}$ in line (K) will be in its lowest terms, if all the factors common to num^r and den^r have been cancelled at the proper time.

Ex. III. $$\tfrac{7}{8} \text{ of } 7\tfrac{1}{2} \text{ of } \tfrac{4}{15} \text{ of } \tfrac{2}{7} = \frac{\cancel{7}}{\underset{2}{\cancel{8}}} \times \frac{\cancel{15}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{15}} \times \frac{\cancel{2}}{\cancel{7}}.$$

Here there are no factors to be *seen* in the num^r; but we must remember that in reality there have been 4 divisions of the num^r by the factors 7, 15, 4, and 2; and therefore 4 quot^ts, which in this case are each = 1; therefore the real num^r = 1 × 1 × 1 × 1, or 1; so the quot^ts in the den^r are 2, 1, 1, 1, and their product = 2; therefore the reduced fraction has 1 as num^r and 2 as den^r, *i. e.* it is $\frac{1}{2}$.

Exs. 11. Find the value of

1. $\frac{2}{3} \times \frac{4}{5}$ of $\frac{6}{7}$
2. $1\frac{3}{5} \times \frac{4}{27} \times 2\frac{7}{9}$ of $\frac{18}{49}$
3. $3\frac{2}{7}$ of $5\frac{4}{9}$ of $\frac{15}{23}$
4. $10\frac{2}{3} \times \frac{17}{24}$ of $\frac{15}{19}$ of $\frac{18}{34}$
5. $2\frac{6}{7}$ of $\frac{14}{15}$ of $5\frac{1}{3}$ of $7\frac{7}{8}$
6. $\frac{1}{2}$ of $\frac{1}{7}$ of $\frac{1}{9}$ of $4\frac{1}{2}$
7. $(3\frac{2}{5} \times 5\frac{1}{2} \times \frac{7}{9}) - (\frac{1}{3}$ of $\frac{5}{12})$
8. $1 - (\frac{3}{11}$ of $\frac{5}{9}$ of $\frac{18}{25})$

DIVISION.

58. The operation of Division may be expressed in two ways;

$$\text{Thus: } \frac{11}{12} \div \frac{5}{9}, \text{ and } \frac{\frac{11}{12}}{\frac{5}{9}}$$

have the same meaning, both being read $\frac{11}{12}$ divived by $\frac{5}{9}$.

We have now to inquire how to perform this division.

We know by (31) that $5 = 9 \times \frac{5}{9} = 9$ times $\frac{5}{9}$, or that 5 is 9 times as large as $\frac{5}{9}$. If, now, I divide $\frac{11}{12}$ by 5 (when my object was to divide by $\frac{5}{9}$), I shall be dividing by a quantity 9 times too large, and therefore my quotient will be 9 times too small; to obtain, then, the *true* quotient, I must multiply the first quot^t by 9; *i. e.* the correct quot^t is found by performing two operations—first, by dividing by 5, and secondly, by multiplying by 9; and performing these two operations at one step, we have

$$\frac{11}{12} \div \frac{5}{9} = \frac{11}{12} \times \frac{9}{5},$$

and by comparing the two sides of this equation, we see that the divisor $\frac{5}{9}$ has been inverted into $\frac{9}{5}$, and and the (÷) has been changed into (×), or the div^n changed into multiplication.

If mixed or whole numbers are found in any expression which is to be simplified, we must, of course, reduce to imp^r fr^{ns}, and invert the divisor, as before. Thus

$$\frac{9}{14\frac{5}{8}} = \frac{\frac{9}{1}}{\frac{117}{8}} = \frac{9}{1} \times \frac{8}{117} = \frac{\cancel{9}}{1} \times \frac{8}{\underset{13}{\cancel{117}}} = \frac{8}{13}.$$

59. The following is a more complicated Ex. involving both Mult^n and Division.

Reduce to the simplest form $\dfrac{7\frac{3}{5}}{8\frac{5}{9}}$ of $\dfrac{6\frac{5}{12}}{8\frac{5}{6}}$.

Here we reduce mixed numbers to imp^r fr^{ns}—change *of* into (×), and invert the divisors, as was shown in the last art.; and then, making two steps in the operation,

$$\text{the expression} = \frac{\frac{38}{5}}{\frac{77}{9}} \times \frac{\frac{77}{12}}{\frac{53}{6}} = \frac{38}{5} \times \frac{9}{77} \times \frac{77}{12} \times \frac{6}{53} \qquad \text{(L)}$$

$$= \frac{\overset{19}{\cancel{38}}}{5} \times \frac{9}{\cancel{77}} \times \frac{\cancel{77}}{\underset{2}{\cancel{12}}} \times \frac{\cancel{6}}{53} \qquad \text{(M)}$$

$$= \frac{19 \times 9}{5 \times 53} = \frac{171}{265}.$$

In working such Exs. I should not write down both (L) and (M), but perform the cancelling on the line (L), which would then become (M): but a pupil would have been confused at first, if he had not seen both lines written in full.

Exs. 12. Find the value of

1. $\frac{11}{12} \div \frac{2}{3}$
2. $\frac{3}{14} \div \frac{1}{7}$
3. $9 \div \frac{4}{5}$
4. $18\frac{1}{2} \div \frac{6}{7}$
5. $\dfrac{15\frac{3}{4}}{9}$
6. $\dfrac{11}{17\frac{8}{11}}$
7. $\frac{1}{2}$ of $\dfrac{3\frac{2}{3}}{16\frac{1}{2}}$
8. $\dfrac{18}{\frac{3}{5} \text{ of } 4\frac{2}{7}} \div 17\frac{1}{2}$
9. $\dfrac{1}{2} + \dfrac{1}{3 + \dfrac{1}{1\frac{1}{2}}}$
10. $\dfrac{1\frac{1}{12}}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}$

SIMPLIFICATION OF FRACTIONS.

60. Under this head is classed a variety of *expressions* intended to exemplify the operations previously illustrated; and the working of the examples under it depends entirely upon a correct and ready use of the methods of performing Addition, Subtraction, &c., as already explained. There is no difficulty in any of them, so long as a pupil will pay close attention to the *signs;* but I will point out one or two Exs.

wherein he is liable to err, though every such error must arise entirely from want of proper observation.

Ex. $4\frac{4}{11}$ of $7\frac{5}{16} + 2\frac{1}{2}$ of 11.

In working this, a pupil must take care that he reduces the whole expression on each side of the (+) to the simplest form, before he begins to perform the operation of Add^n, which that sign (+) bids him perform. The same caution would be necessary if there had been (–) instead of (+) in the above example.

I will work this Ex. as it ought to be written; and leave a large space on each side of the (+), so that I may run no risk of connecting the two parts incorrectly.

$$4\tfrac{4}{11} \text{ of } 7\tfrac{5}{16} \quad + \quad 2\tfrac{1}{2} \text{ of } 11$$

$$= \frac{\overset{3}{\cancel{48}}}{11} \times \frac{117}{\cancel{16}} \quad + \quad \frac{5}{2} \times \frac{11}{1}$$

$$= \qquad \frac{351}{11} \quad + \quad \frac{55}{2} \qquad\qquad [\text{L. C. D.} = 22]$$

$$= \qquad \frac{702 \quad + \quad 605}{22} = \frac{1307}{22} = 59\tfrac{9}{22}.$$

Ex. II. Simplify $\dfrac{3\frac{1}{2} \text{ of } \frac{5}{8}}{2\frac{5}{6} \text{ of } 3\frac{1}{3}} \div \dfrac{2\frac{2}{11} \text{ of } \frac{11}{12}}{3\frac{1}{9} \text{ of } 7\frac{5}{7}}$.

Now, the whole of the fr^n to the right of the sign (÷) is a divisor of the fr^n to the left of the sign; therefore this right-hand fr^n must be inverted, and the sign (÷) changed to (×): the expression will then be

$$\frac{3\frac{1}{2} \text{ of } \frac{5}{8}}{2\frac{5}{6} \text{ of } 3\frac{1}{3}} \times \frac{3\frac{1}{9} \text{ of } 7\frac{5}{7}}{2\frac{2}{11} \text{ of } \frac{11}{12}};$$

and this, by changing *of* into ×, and the mixed numbers into imp^r fr^{ns}, becomes

$$\frac{\dfrac{7}{2} \times \dfrac{5}{8}}{\dfrac{17}{6} \times \dfrac{10}{3}} \times \frac{\dfrac{28}{9} \times \dfrac{54}{7}}{\dfrac{24}{11} \times \dfrac{11}{12}},$$

and lastly,—inverting the fr^{ns} in the lower line, because they are divisors, and connecting all the fr^{ns} with mult^n signs, I have

$$\text{the entire expression} = \frac{7}{2}\times\frac{5}{8}\times\frac{6}{17}\times\frac{3}{10}\times\frac{28}{9}\times\frac{54}{7}\times\frac{11}{24}\times\frac{12}{11} \quad \text{(N)}$$

$$= \frac{\cancel{7}}{\cancel{2}}\times\frac{\cancel{5}}{\underset{4}{\cancel{8}}}\times\frac{\overset{3}{\cancel{6}}}{17}\times\frac{3}{\underset{\cancel{2}}{\cancel{10}}}\times\frac{\overset{\overset{7}{\cancel{14}}}{\cancel{28}}}{\cancel{9}}\times\frac{\overset{\overset{3}{\cancel{6}}}{\cancel{54}}}{\cancel{7}}\times\frac{\cancel{11}}{\underset{\cancel{2}}{\cancel{24}}}\times\frac{\cancel{12}}{\cancel{11}}\,{}^{*}$$

$$= \frac{3\times3\times7\times3}{4\times17} = \frac{189}{68} = 2\tfrac{53}{68}.$$

In working such Exs. I should omit line (N) for the reason given in (59).

I subjoin an Ex., wherein the whole quantity enclosed in ( ) parentheses, or brackets, as they are called, must be reduced to as simple a form as possible, before the (×) or (÷) joining the expressions in brackets can be made use of.

Thus; if I have to simplify $(\frac{7}{8}-\frac{3}{10})\times(2\frac{3}{4}+3\frac{1}{9})$, I shall first reduce $\frac{7}{8}-\frac{3}{10}$ to the simplest form; then $2\frac{3}{4}+3\frac{1}{9}$ must be reduced in like manner, and these two results must then be multiplied together, because the sign (×) is placed between the brackets. The whole work will stand thus;

$$\frac{7}{8}-\frac{3}{10}=\frac{35-12}{40}=\frac{23}{40} \qquad \text{I.} \qquad (\text{since L. C. D.}=40)$$

$$\text{and } 2\tfrac{3}{4}+3\tfrac{1}{9}=\frac{11}{4}+\frac{28}{9}=\frac{99+112}{36}=\frac{211}{36} \qquad \text{II.}$$

Therefore, multiplying I. and II., I have

$$\left(\frac{7}{8}-\frac{3}{10}\right)\times\left(2\tfrac{3}{4}+3\tfrac{1}{9}\right)=\frac{23}{40}\times\frac{211}{36}$$

$$=\frac{4853}{1440}=3\tfrac{533}{1440}.$$

Exs. 13. Reduce to the simplest forms

I. $\frac{3}{11}$ of $8\frac{2}{3}-\frac{5}{7}$; $\frac{3}{11}$ of $(8\frac{2}{3}-\frac{5}{7})$; $\frac{3}{11}\sim(8\frac{2}{3}$ of $\frac{5}{7})$

II. $\frac{2}{3}$ of $\frac{6}{7}\div\frac{8}{9}$ of $17\frac{1}{2}$; $\frac{8}{17}+(\frac{3}{5}$ of $7\frac{1}{2})\div\frac{9}{11}$; $(\frac{8}{17}+\frac{3}{5}$ of $7\frac{1}{2})\div\frac{9}{11}$

III. $\dfrac{2\frac{2}{3}\times(3\frac{1}{2}+\frac{7}{9})}{\frac{3}{5}\sim\frac{2}{3}}$; $\dfrac{7\frac{1}{2}\times3\frac{1}{3}}{6\frac{1}{2}\text{ of }\frac{11}{13}}\div\dfrac{\frac{5}{8}\text{ of }11}{\frac{1}{7}}$

IV. $\dfrac{\frac{2}{5}}{\frac{8}{9}}+\dfrac{\frac{5}{7}}{7\frac{1}{2}\text{ of }\frac{11}{12}}$; $(5\frac{2}{3}+3\frac{1}{4})\times(6\frac{1}{7}$ of $\frac{1}{56})$.

---

* A pupil will not necessarily perform the cancelling in exactly the same order as above; but the result will be the same.

REDUCTION OF FRACTIONS.

Def. Any quantity involving money, weight, &c.,—as 4 shillings, 5 pence, $\frac{3}{4}$oz., is called a *concrete* quantity; and any number not involving any such denominations,—as 6, 8, $\frac{4}{5}$, is called an *abstract* number.

61. It is here intended to express in positive terms such quantities as $\frac{5}{8}s.$, $\frac{11}{12}$ of 10*s.* 6*d.* &c.; *i. e.* to express concrete quantities, being frac^{l} parts of any given den^{n}, in terms of lower den^{ns}.

Thus, $\frac{5}{8}s.$ must be expressed in terms of pence, and frac^{l} parts of a penny; $\frac{11}{12}$ of 10*s.* 6*d.* in terms of shillings, pence, and frac^{l} parts of 1*d.*

So, also, $\frac{9}{11}$ of a ton would be expressed in cwts., qrs., lbs., oz., drs., and frac^{l} parts of a dram.

It may happen that the proposed fr^{n} can be expressed in an exact number of units of some one of the lower den^{ns}; then, of course, there will be no frac^{l} part of the last named den^{n}.

62. I will now show how to perform the operation intended in the last article.

It has been seen (21) that $\frac{5}{8}s.$ means either *one*-eighth of 5 shillings, or *five*-eighths of one shilling. We may, therefore, use either of these two methods:—divide the 5*s.* by 8, as in Comp^{d} Short Division, and the process will be thus—

$$\begin{array}{r} 8\,)\,\underline{5\quad 0\;} \\ \underline{\underline{7\tfrac{1}{2}d.}} \end{array}$$

or, treating the fr^{n} as $\frac{5}{8}$ of 1*s.*, we may change *of* into (×), reduce the 1*s.* to pence, and proceed thus—

$$\frac{5}{8}\text{ of }1s. = \frac{5\times 12}{8}d. = \frac{5\times \overset{3}{\cancel{12}}}{\underset{2}{\cancel{8}}}d. = \frac{15}{2}d. = 7\tfrac{1}{2}d.$$

This latter method seems more suitable to an Ex. which professes to belong to fr^{ns}, and is the one which I recommend, unless the den^{r} of the proposed fr^{n} be very large, so as to make the division to be performed rather difficult: in that case, to work by Comp^{d} Long Div^{n} is preferable, as being more likely to be correct. I will take such an Ex. and work it through by both methods.

Express in positive terms $\frac{17}{26}$ of a ton.

Working fractionally, we have

$$\frac{17}{26} \text{ of } 1 \text{ ton} = \frac{17 \times \overset{10}{\cancel{20}}}{\underset{13}{\cancel{26}}} \text{ cwts.} = \frac{170}{13} \text{ cwts.} = 13\tfrac{1}{13} \text{ cwts.}$$

I now take the frac^{l} part of the cwts., viz. $\frac{1}{13}$ of 1 cwt., and express it in the next lower den^{n}, viz. quarters, and so proceed, till either there be no frac^{l} part left, or till I come to the last den^{n} in Avoirdupois weight.

$\frac{1}{13}$ of 1 cwt. $= \frac{1 \times 4}{13}$ qrs. $= \frac{4}{13}$ qrs. (*i. e.* 0 qrs. in the final quotient.)

$\frac{4}{13}$ of 1 qr. $= \frac{4 \times 28}{13}$ lbs. $= \frac{112}{13}$ lbs. $= 8\frac{8}{13}$ lbs.

$\frac{8}{13}$ of 1 lb. $= \frac{8 \times 16}{13}$ oz. $= \frac{128}{13}$ oz. $= 9\frac{11}{13}$ oz.

$\frac{11}{13}$ of 1 oz. $= \frac{11 \times 16}{13}$ drs. $= \frac{176}{13}$ drs. $= 13\frac{7}{13}$ drs.

therefore, $\frac{17}{26}$ of 1 ton $= \overset{\text{cwts.}}{13}\ \ \overset{\text{qrs.}}{0}\ \ \overset{\text{lbs.}}{8}\ \ \overset{\text{oz.}}{9}\ \ \overset{\text{drs.}}{13\frac{7}{13}}$.

Secondly, take $\frac{1}{26}$ of 17 tons; *i. e.* divide 17 tons by 26.

```
      tons.
       17
        20
26) 340  (13 cwt.
     26
      80
      78
       2
```

$$\begin{array}{r@{\,}r@{\;}l}
 & 2 & \\
 & \underline{4} & \\
26) & 8 & (0\text{ qrs.} \\
 & \underline{28} & \\
26) & 224 & (8\text{ lbs.} \\
 & \underline{208} & \\
 & 16 & \\
 & \underline{16} & \\
26) & 256 & (9\text{ oz.} \\
 & \underline{234} & \\
 & 22 & \\
 & \underline{16} & \\
26) & 352 & (13\tfrac{7}{13}\text{ drs.} \\
 & \underline{26} & \\
 & 92 & \\
 & \underline{78} & \\
 & \dfrac{14}{26} & = \dfrac{7}{13}
\end{array}$$

therefore, as before, $\frac{17}{26}\text{ tons} = \overset{\text{cwts.}}{13}\ \overset{\text{qrs.}}{0}\ \overset{\text{lbs.}}{8}\ \overset{\text{oz.}}{9}\ \overset{\text{drs.}}{13\tfrac{7}{13}}$.

63. When Exs. under this Case involve money, it is well to carry the reduction no farther than pence and frac^{l} parts of a penny: for, since farthings are themselves written as frac^{l} parts of a penny, therefore if we carry the reduction to frac^{l} parts of a farthing, the answer will appear somewhat complicated. I will work an Ex. which will illustrate this.

Express in positive terms $\frac{151}{240}s.$

$$\frac{151}{240}s. = \frac{151 \times \cancel{12}}{\underset{20}{\cancel{240}}}d. = \frac{151}{20}d. = 7\tfrac{11}{20}d. \qquad \text{(O)}$$

$$\frac{11}{20}d. = \frac{11 \times \cancel{4}}{\underset{5}{\cancel{20}}}f. \quad = \frac{11}{5}f. = 2\tfrac{1}{5}f.$$

$$\text{therefore } \frac{151}{240}s. = 7\tfrac{1}{2}d.\ \tfrac{1}{5}f. \qquad \text{(P)}$$

If I compare $7\frac{11}{20}d.$ in (O) and $7\frac{1}{2}d.\ \frac{1}{5}f.$ in (P), I see that the former fr^{n} involves less labour, and conveys to a person acquainted with fr^{ns} the idea of the real value of the expression, at least as well as the other.

64. When a unit of any den^n can be divided exactly by any number, as 4, 5, 6, &c., such fourths, fifths, sixths, &c., are called *aliquot* parts of that unit.

Thus, since £1 or 20*s.* when divided by 3, 4, 5, 6, 8, 10, 12, gives quotients involving an exact number of shillings and pence; therefore thirds, fourths, fifths, &c., of £1 are called aliquot parts of £1.

For example

$\frac{1}{5}$£ = 4*s.* $\frac{2}{5}$£ = 8*s.* $\frac{4}{5}$£ = 16*s.*

$\frac{1}{8}$£ = 2*s.* 6*d.* $\frac{5}{8}$£ = 12*s.* 6*d.* $\frac{7}{8}$£ = 17*s.* 6*d.*

$\frac{1}{3}$£ = 6*s.* 8*d.* $\frac{2}{3}$£ = 13*s.* 4*d.*

So also, $\frac{1}{8}$*s.* = $1\frac{1}{2}$*d.* $\frac{5}{8}$*s.* = $7\frac{1}{2}$*d.* $\frac{7}{8}$*s.* = $10\frac{1}{2}$*d.*

And since 1 lb. avoirdupois, or 16 oz., can be divided exactly by 2, 4, 8; therefore halves, fourths, and eighths of 1 lb. can be at once expressed in oz.

Thus, $\frac{3}{8}$lb. = 6 oz. $\frac{7}{8}$lb. = 14 oz.

and if the value of one or more of these aliquot parts of the denominations in common use be remembered, the operations performed in this Case will often be much shortened.

$$\text{Ex. } \frac{27}{160}£ = \frac{27\times\cancel{20}}{\underset{8}{\cancel{160}}}s. = \frac{27}{8}s. = 3\tfrac{3}{8}s. = 3s.\ 4\tfrac{1}{2}d.$$

Exs. 14. Express in positive terms

I. $\frac{2}{3}$ of 10*s.*; $\frac{7}{8}$ of 27*s.*; $\frac{3}{7}$ of $\frac{4}{15}$ of 19*s.* 6*d.*

II. $\frac{9}{11}$ of 1 ton; $\frac{5}{16}$ of a cub. ft.; $\frac{7}{15}$ of a quarter of corn.

III. $\frac{5\frac{1}{3}}{7\frac{1}{2}}$ of $\frac{7}{8}$ of a hhd. of wine; $\frac{5}{11}$ of 1 week; $\frac{7}{15}$ of 365$\frac{1}{4}$ days.

IV. $\frac{3}{8}$ of $\frac{5}{9}$ of a Fr. ell; $\frac{1}{11}$ of $\frac{9}{11}$ of $\frac{1}{7\frac{1}{2}}$ of a square mile.

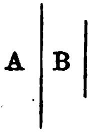
V. $\frac{7}{15}$ of a sq. yd.; $2\frac{1}{2}$ of $\frac{5}{9}$ of a lb. (Apoth.)

RATIO AND PROPORTION.

65. The next operation in Fractions will consist of the expression of one quantity in terms of, or as a frac^{l} part of, another of like nature. But before proceeding to attempt this operation, it will be advisable to discuss one of the most important relations in numbers, without which a pupil cannot understand the principle upon which such Exs. will be worked. This relation is termed RATIO.

And as this consideration of Ratio leads to the doctrine of PROPORTION, I have thought it well to continue the subject in one unbroken thread, and then refer to this explanation, when I come to the treatment of questions involving Proportion and its applications.

66. When two quantities are placed before us—as, for instance, the lines A, B—and we wish to compare their magnitude, we perceive that there is a certain relation between them, which we familiarly call the comparative size of them; and this comparison or relation is measured by seeing how often the one quantity contains the other.



Now, to see how often one quan^{y} contains the other—as, for example, how often 36 contains 5—we divide the former by the latter, *i. e.* the 36 by the 5, and the quotient is $\frac{36}{5}$; therefore, to see how often the line A contains the line B, I divide A by B: *i. e.* the relation of A to B is expressed by the fr^{n} $\frac{A}{B}$.

This relation is called the *ratio* of A to B, and is thus expressed, A : B.

If $A > B$, the fr^{n} expressing the ratio of A to B, *i. e.* $\frac{A}{B}$, will be an improper fr^{n} greater than 1;

If $A = B$, the fr^n will become $\frac{A}{A}$, or unity;

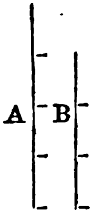
If $A < B$, the fr^n will become $\frac{A}{B}$, a proper fr^n.

Also, when the fr^n $\frac{A}{B}$ is greater than 1, the ratio of A to B is called a ratio of *greater* inequality, because $A > B$.

When the fr^n $\frac{A}{B} = \frac{A}{A} = 1$, the ratio is called a ratio of equality; and when the fr^n $\frac{A}{B}$ is a proper fr^n, the ratio is called a ratio of *less* inequality, because $A < B$.

67. If the num^r and den^r of a fr^n expressing ratio both consist of the same kind of *concrete* quantities, the ratio between these quantities will be an *abstract* number.

Ex. If the lines A and B represent 4 inches and 3 inches respectively, then $A : B = \frac{A}{B} = \frac{4 \text{ in.}}{3 \text{ in.}} = \frac{4}{3}$; and if, instead of inches, these lines had represented 4 feet and 3 feet, or 4 lb. and 3 lb., the ratio of A to B would still be $\frac{4}{3}$, and of B to A would be $\frac{3}{4}$.



But if the num^r and den^r be of the same species of concrete quantity, but not expressed in the same denomination, the ratio cannot be represented by an abstract quantity, until they both be reduced to the same denomination. Thus, $\frac{4 \text{ feet}}{9 \text{ inches}}$ does not $= \frac{4}{9}$ feet, or $\frac{4}{9}$ in., or $\frac{4}{9}$, but it $= \frac{48 \text{ in.}}{9 \text{ in.}} = \frac{48}{9} = \frac{16}{3}$, an abstract number.

If quantities be not of the same nature, there can be no ratio between them; thus $\frac{4 \text{ feet}}{5 \text{ minutes}}$ is no ratio at all, since we cannot compare the magnitude of 4 feet and of 5 minutes.

This article must be especially noticed, as upon it will depend the mode of working the Exs. which have been alluded to in (65).

68. We now see that any fraction, as $\frac{2}{5}$, has still a further meaning, beside what was stated in (21), viz. *the ratio of 2 to 5*, or, as it is written, 2 : 5; and $\frac{2}{5}$ may be called *a ratio.*

If, then, two other quantities be taken, as 4 and 10, and it is found that $\frac{4}{10}=\frac{2}{5}$, then we see that the ratio of 4 to 10 is equal to the ratio of 2 to 5, and this fact we express either thus,

$$4 : 10 = 2 : 5, \quad \text{or } 4 : 10 :: 2 : 5;$$

the latter expression is thus read; 4 is to 10 as 2 is to 5.

When, therefore, two ratios, as $\frac{4}{10}$ and $\frac{2}{5}$, are placed before us, and we learn that they are equal, we say that the quantities, 4, 10, 2, 5, are *Proportionals.*

69. We may now define Proportion to consist in the equality of two ratios. Of the above-mentioned numbers 4, 10, 2, 5; 4 and 5 are called the *extremes;* 10 and 2 are called the *means*, because they are intermediate between the extremes.

Also, when 4 quantities are given, and we wish to ascertain whether they are proportionals or not, we must see if the fr$^{\text{n}}$ expressing the ratio of the 1st and 2nd = the fr$^{\text{n}}$ expressing the ratio of the 3rd and 4th. Thus, if I take the numbers 5, 6, 7, 8, and wish to try whether they are proportionals, I compare $\frac{5}{6}$, the ratio of the first pair, and $\frac{7}{8}$, the ratio of the second pair; and if these fr$^{\text{ns}}$ are proved unequal, the above 4 quantities are not proportionals.

Now, it was shown in (50), that to compare two frac^l quantities we must bring them to some Com. Den^r. Reducing the fr^{ns} $\frac{5}{6}$ and $\frac{7}{8}$ to a C. D. 48, we have to compare

$$\frac{5 \times 8}{48} \text{ and } \frac{6 \times 7}{48} \qquad (Q)$$

And since $\frac{40}{48}$ is not $= \frac{42}{48}$, therefore $\frac{5}{6}$ is not $= \frac{7}{8}$, and 5, 6, 7, 8, are not proportionals.

70. It will be seen in line (Q) that the test as to whether or not the four proposed quantities be proportionals, consists in multiplying, 1st, the two extremes 5, 8, and 2ndly, the two means 6, 7: and if these products be equal, the four quantities are proportionals; if they be not equal, the four quantities are not proportionals.

This operation may generally be performed *mentally:* thus, if I take 5, 8, 9, 16, I say $5 \times 16 = 80$, and $8 \times 9 = 72$; therefore these quantities are not proportionals. Again, if I take 4, 9, 5, $11\frac{1}{4}$, since the product of the extremes, $4 \times 11\frac{1}{4} = 4 \times \frac{45}{4} = 45$, and the product of the means, $9 \times 5 = 45$; therefore these last four quantities are proportionals. (See Appendix, Art. Ratio.)

71. Having now explained how to tell when four quantities are proportionals, we proceed to show how, when *three* numbers are given, we may find a fourth number, such that the other three and this fourth shall be proportionals; that is, to solve the following question:

If three numbers be given, as 5, 6, 10, what must be taken as a fourth number, such that the four, when taken in order, shall be proportionals?

Now, as I do not know (or, at least, the learner does not know), the required number, let the letter N stand for or represent this number; and I have now to try and find what this N must be.

Since, then, 5, 6, 10, and N are required to be proportionals, therefore we must have the ratio between the first pair 5, 6, = that between the second pair 10, N; *i. e.*

$$\frac{5}{6} = \frac{10}{N}; \qquad \text{(R)}$$

Multiplying both sides by N, according to (28), we have $\frac{N \times 5}{6} = \frac{10}{1} = 10$; next, multiplying both sides by 6, $N \times 5 = 10 \times 6$; lastly, dividing by 5, according to (29), we obtain

$$N = \frac{6 \times 10}{5} \qquad \text{(S)}$$

$$= \frac{6 \times \overset{2}{\cancel{10}}}{\cancel{5}} = 12.$$

Put 12 for N in (R), and we have $\frac{5}{6} = \frac{10}{12}$, which we know to be true, and therefore 5, 6, 10, 12, are proportionals[*].

72. The result of this article must be most carefully noticed; for in observing (S) we learn that the required 4th number N was thus formed from the three former ones, 5, 6, 10: viz. that it is the value of this fr^n—the product of the 2nd and 3rd terms, divided by the first; or it $= \frac{\text{2nd}}{\text{1st}} \times \text{3rd}$.

73. By observing the successive steps whereby N was obtained, we learn that when two fractions are equal, a factor of one den^r, or the entire den^r, may be transferred into the opposite num^r, and conversely, the num^r, or any factor of it, may be transferred into the opposite den^r, without disturbing the equality. Hence also, when two fractions are equal,

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[*] Or, since the product of the extremes N, 5 = that of the means 6, 10; I have $N \times 5 = 6 \times 10$; $\therefore N = \frac{6 \times 10}{5}$, as before.

they will be equal when inverted. A quick use of these properties is very serviceable in the solution of Algebraic Equations.

Exs. 15. Find a fourth proportional to each of the following sets of numbers:—

I. 8, 7, 15; 5, 2, 11; 9, 8, $\frac{1}{7}$;
II. $3\frac{1}{2}$, $7\frac{1}{4}$, $\frac{1}{6}$; $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{4}$; $\frac{1}{5}$, $8\frac{1}{2}$, $\frac{1}{10}$;
III. 19, 18, 17; 17, 18, 19.

74. We have shown that, in order that there may exist a ratio between two quantities, they must be of the same kind; since, then, there is to be a proportion among the three given quantities and the fourth, therefore between the 1st and 2nd there must be a ratio, and an equal ratio between the 3rd and 4th. Hence we see that the second pair must be of the same kind; that is, the 4th quantity, when found, ought to be of the same kind as the 3rd. Now, if we observe the frⁿ $\frac{\text{2nd}}{\text{1st}} \times \text{3rd}$ in the last article, we notice, that since it is allowed that the 2nd and 1st are alike in kind, therefore the $\frac{\text{2nd}}{\text{1st}}$ will be an abstract number (67); and therefore the 4th, which $= \frac{\text{2nd}}{\text{1st}} \times \text{3rd}, =$ 3rd multiplied by some abstract number; *i. e.* it will be the 3rd term repeated as many times as there are units in this number, and therefore will be of the same kind as that in which the third term was expressed.

Also, if the 2nd term be greater than the 1st, $\frac{\text{2nd}}{\text{1st}}$ will be a frⁿ > 1, and this multiplier of the 3rd term will be greater than 1, and the 4th term greater than the 3rd; but if the 2nd term be less than the 1st, then $\frac{\text{2nd}}{\text{1st}}$ will be a proper frⁿ, and the multiplier less than 1, and the 4th term less than the 3rd. (See Appendix, Art. *Proportion*.)

75. We may here notice a form of expression that we familiarly use, when we say, that two quantities are proportional to one another; as, for instance, that the amount of a servant's wages is proportional to the length of his service. Here, apparently, two quantities of different kinds are compared, viz. money and time; but, in reality, four quantities are implied, viz. two periods of time, and two amounts of wages. For since one year and one year's wages may be taken as fixed standards, by which we may measure other periods and other amounts; therefore I mentally compare any proposed length of service and the corresponding amount of wages with these fixed standards; and the original expression means this—that there is the same ratio between any given length of service and one year, that there is between the amount of wages for that service and one year's wages; or, that these two periods of service and two amounts of wages together form a proportion in this order,

Given time : 1 year :: wages for that time : wages for 1 year.

If we wish to ascertain whether two quantities are what is here called proportional to one another, according to the common usage of the words, we may try if doubling the one quantity causes the other to be doubled; as for example, in this instance—will the wages be doubled, if the time of service be doubled? Since this is the case, therefore the wages and time are proportional in the sense explained above.

The correct, though not common mode of expressing this proportion, is to say, that the amount of wages *varies as* the length of the time of service.

76. But sometimes this proportion occurs in a different form; as, when a man has a certain number of miles to walk in any time; then, if he quickens his rate of going, he can complete his task in less time. In this case it would be incorrect to say that the length of time was in proportion to the rate of walking; for, according to the test given above, I ask—if the rate be doubled, will the number of days be *doubled?* the answer is, no: on the contrary, it will be *halved.* There is here, then, evidently a proportion (taking the same word in the common usage,) but of a different kind, viz. that as one number is increased by multiplication, the other is correspondingly diminished by division.

As, in the first instance, it was said that the wages *varied as* the time, so, in this second example, the relation between the two quantities is correctly expressed by saying, that the number of days *varies inversely as* the rate of walking; or, as it is commonly said—in the first Ex., the wages were *directly* proportional to the time; and in the second Ex., the number of days is *inversely* proportional to the rate.

Proportion has here been shown to involve 4 quantities; these need not, however, be all different. Thus, if I find 3 quantities, whereof the ratio between the 1st and 2nd = that between the 2nd and 3rd, then the three quantities are said to be proportional: thus, 3, 6, 12 are proportionals, because $\frac{3}{6} = \frac{6}{12}$, or $3 : 6 = 6 : 12$ (T)

Also, to find this 3rd propor^{l} 12, when the two former ones are given, we must compare line (T) with an ordinary proportion containing 4 quan^{s}, and we shall see that 12, the required 3rd, is in the 4th place, and the 2nd quan^{y}, 6, is repeated in the 3rd place: hence, the usual equation $4\text{th} = \frac{2\text{nd}}{1\text{st}} \times 3\text{rd}$, will become $3\text{rd} = \frac{2\text{nd}}{1\text{st}} \times 2\text{nd}$; and if 3 and 6 be the 1st and 2nd terms, we have $3\text{rd} = \frac{6}{3} \times 6 = 12$.

Exs. 16.

Find a third proportional to each of the following pairs of numbers :—

iv. 8, 12 ; 12, 18 ; 15, 16 ; 20, 15 ;

v. $\frac{1}{3}$, $\frac{1}{4}$; $2\frac{1}{3}$, $\frac{2}{5}$; $\frac{1}{2}$ of $\frac{2}{3}$, $\frac{2}{7}$ of 3.

REDUCTION OF FRACTIONS.

77. We are now able to perform the operation alluded to in (65) viz. to express one quantity in terms of another, or as a frac^{l} part of another. This fr^{n} connecting the two quantities will show the ratio between them, and may be either proper or improper.

For example, to express 9*d.* in terms of £1, is to see what ratio 9*d.* bears to £1. Hence, according to what has been explained concerning ratio, we say,

$$9d. \text{ in terms of } £1 = 9d. : £1.$$

$$= \frac{9d.}{£1} = \frac{9d.}{240d.} = \frac{9}{240} = \frac{3}{80};$$

and transferring the £1 from the den^{r} of the left-hand fr^{n} to the num^{r} of the right-hand fr^{n} by (73), I have $9d. = \frac{3}{80} \times £1$, or $\frac{3}{80}$ of £1. This result may also be obtained as follows :—

I have to express pence in terms of £1.

Now, 240 pence = £1 ; therefore, taking $\frac{1}{240}$th part of both sides—

$$1 \text{ penny} = \frac{1}{240} \text{ of } £1;$$

and multiplying by 9,

$$9 \text{ pence} = \frac{9}{240} \text{ of } £1;$$

$$= \frac{3}{80} \text{ of } £1;$$

and transferring the £1 from the right-hand numerator to the left-hand denominator, we have, as before,

$$\frac{9 \text{ pence}}{£1} = \frac{3}{80}$$

It will be observed, that by this second method we have an independent proof, that the ratio between two concrete numbers of like kind, as pence, pounds, &c., is an abstract number.

78. Either of the two methods just shown may be employed; but I prefer the former, because it more decidedly keeps the idea of ratio before the mind; and an Ex. so worked can be written out in a more condensed form than by the latter method. It will be noticed that the two concrete quantities are *both* to be reduced to some com^n den^n, in order that the ratio between them may be expressed: thus, in the last Ex., both were reduced to pence.

Ex. II. Express 7*s.* 7½*d.* in terms of 10*s.* 6*d.*

Here I exhibit both numerator and denominator in terms of pence; and write

$$\frac{7s.\ 7\tfrac{1}{2}d.}{10s.\ 6d.} = \frac{91\tfrac{1}{2}d.}{126d.} = \frac{\frac{183}{2}}{126} = \frac{\overset{61}{\cancel{183}}}{2 \times \underset{42}{\cancel{126}}} = \frac{61}{84}$$

$$\text{or,} = \frac{7\tfrac{5}{8}s.}{10\tfrac{1}{2}s.} = \frac{61}{\underset{4}{\cancel{8}}} \times \frac{\cancel{2}}{21} = \frac{61}{84}$$

I might have reduced both numerator and denominator to half-pence, but a pupil would not easily reduce 7*s.* 7½*d.* to half-pence mentally; whereas, by both the above methods, the whole work is fully shown.

Referring to Ex. I., we read—$\frac{9d.}{£1} = \frac{3}{80}$; and since, by (73), two equal fractions may be inverted, without disturbing the equality; therefore $\frac{£1}{9d.} = \frac{80}{3}$; or, changing the 9*d.* to the right-hand numerator,

$$£1 = \frac{80}{3} \times 9d.;\ \text{or} = \frac{80}{3} \text{ of } 9d.$$

Hence we see that in every Ex. where we have to express one quantity in terms of a second, we can express this second in terms of the first, by merely inverting the ratio which connected the first and second. Thus, in Ex. II.,

$$\text{since } 7s.\ 7\tfrac{1}{2}d. = \frac{61}{84} \text{ of } 10s.\ 6d.\,;$$

$$\text{therefore } 10s.\ 6d. = \frac{84}{61} \text{ of } 7s.\ 7\tfrac{1}{2}d.$$

79. Sometimes fractions may be involved in both num^{r} and den^{r}, as in

Ex. III. Express $1\frac{3}{4}$ of 2a. 1r. in terms of 3 acres $2\frac{1}{2}$ roods.

Here, bringing both quantities into roods, and changing *of* into ×, we have

$$\frac{1\frac{3}{4} \text{ of 2a. 1r.}}{\text{3a. } 2\frac{1}{2}\text{r.}} = \frac{\frac{7}{4} \times \text{9r.}}{14\frac{1}{2}\text{r.}} = \frac{\frac{7}{4} \times 9}{\frac{29}{2}}$$

$$= \frac{7}{\underset{2}{\cancel{4}}} \times \frac{9}{1} \times \frac{\cancel{2}}{29} = \frac{63}{58}$$

This should be left as an improper fraction, because I can then read off this desired result—that

$$1\tfrac{3}{4} \text{ of 2a. 1r.} : \text{3a. } 2\tfrac{1}{2}\text{r.} = 63 : 58\,;$$

or that the ratio of these two portions of land is that of 63 to 58.

As in (64) it was shown to be advisable to be able to express aliquot parts of any den^{n} in terms of lower den^{ns}, so, with a little experience, a pupil will, by a reverse process, mentally work easy Exs. in this Case, so as at once to see, that *5s.* in terms of £1 = $\frac{1}{4}$; that *6s. 8d.* : £1 = $\frac{1}{3}$, and so on: and a readiness in performing such simple reductions will often materially shorten the labour of more complicated Exs. Thus, to express $\frac{7}{8}$ of 13*s.* 4*d.* in terms of 10*s.*:—

By observing that 13*s.* 4*d.* = $\frac{2}{3}$£, and 10*s.* = $\frac{1}{2}$£, I have

$$\frac{\frac{7}{8} \text{ of } 13s.\ 4d.}{10s.} = \frac{\frac{7}{8} \text{ of } \frac{2}{3}£}{\frac{1}{2}£} = \frac{7}{\underset{\underset{2}{\cancel{4}}}{\cancel{8}}} \times \frac{\cancel{2}}{3} \times \frac{\cancel{2}}{1} = \frac{7}{6}\,;$$

or, $\frac{7}{8}$ of 13*s.* 4*d.* = $\frac{7}{6}$ of 10*s.*

By the ordinary method I should have reduced both 13*s.* 4*d.* and 10*s.* to pence or fourpences, and the resulting fraction would have required much heavier reduction than mine has needed.

80. When either the num^r or den^r of the left-hand fr^n requires much reduction to a lower den^n, it is better to express by signs the multiplication requisite to perform this reduction, rather than to perform the mult^n, and put down the result; because, when either num^r or den^r is thus expressed in factors, it is in the best form for detecting the probability of any cancelling taking place, so as to present the resulting fr^n in its lowest terms.

For example, in reducing 5drs. 1sc. 15grs. to the fraction of 1lb., I must bring the 1lb. to grains, and I write

$$\frac{5\text{dr. }1\text{sc. }15\text{grs.}}{1\text{lb.}}=\frac{16\text{sc. }15\text{grs.}}{1\times 12\times 8\times 3\text{sc.}}=\frac{335\text{grs.}}{12\times 8\times 3\times 20\text{grs.}}\qquad\text{(U)}$$

$$=\frac{\overset{67}{\cancel{335}}}{12\times 8\times 3\times \underset{4}{\cancel{20}}}=\frac{67}{96\times 12}=\frac{67}{1152}.$$

But in this Ex. we may also notice that the last denomination 15grs. may be very readily expressed as a fraction of the preceding denomination, viz. scruples, for 15grs. $=\frac{15}{20}$sc. $=\frac{3}{4}$sc.; therefore, instead of the two latter fractions in line (U), I should write

$$=\frac{16\frac{3}{4}\text{sc.}}{12\times 8\times 3\text{sc.}}=\frac{\frac{67}{4}}{12\times 8\times 3}=\frac{67}{4\times 12\times 8\times 3}=\frac{67}{12\times 96}$$
$$=\frac{67}{1152}.$$

Of these five fractions the last four are merely reductions to a simpler form; and the number of steps which a pupil may have to write down in working Exs. similar to the one above will depend, partly on the magnitude of the numbers involved, and partly on his own quickness in working mentally.

I will give one more Ex. in which the work is condensed, but still the successive operations are intelligible.

Express $\frac{11}{12}$ of $4\frac{1}{4}d.$ in terms of $9s.\ 7\frac{1}{2}d.$

$$\frac{\frac{11}{12}\text{ of }4\frac{1}{4}d.}{9s.\ 7\frac{1}{2}d.}=\frac{\frac{11}{12}\times\frac{17}{4}d.}{115\frac{1}{2}d.}=\frac{\cancel{11}}{12}\times\frac{17}{\underset{2}{\cancel{4}}}\times\frac{\cancel{2}}{\underset{21}{\cancel{231}}}=\frac{17}{504};$$

$$\text{or, }\frac{11}{12}\text{ of }4\frac{1}{4}d.=\frac{17}{504}\text{ of }9s.\ 7\frac{1}{2}d.$$

Exs. 17.

1. Express 1*s*. 3*d*.,...... and 17*s*. 6*d*...... in terms of £1.
2. ,, 17*s*. 7$\frac{1}{2}$*d*.,... ,, 25*s*. 9$\frac{1}{4}$*d*...... ,, 1 guinea.
3. ,, 12*s*. 7$\frac{5}{7}$*d*.,... ,, 2*s*. 11$\frac{1}{3}$*d*...... ,, 15*s*. 6*d*.
4. ,, $\frac{5}{13}$ of a moidore, ,, 8$\frac{3}{4}$ groats..... ,, 12$\frac{1}{2}$ guineas.
5. ,, $\frac{3}{7}$£,........... ,, $\frac{1}{1\frac{2}{3}}$ of 7*s*....... ,, 1 farthing.
6. ,, 3 qrs. 1$\frac{1}{3}$ nls., ,, $\frac{1}{2}$ in............ ,, 1 Flem. ell.
7. ,, 3 oz. 5 drs. 1$\frac{1}{4}$ sc......................... ,, 1 lb.
8. ,, $\frac{5}{7}$ of 1$\frac{1}{2}$ guineas......................... ,, £5.
9. ,, $\frac{3}{8}$ of 1$\frac{3}{5}$ of an acre....... .... ,, 1 sq. yd.
10. ,, $\frac{10}{11}$ of 1$\frac{5}{6}$ of 3 miles.................. ,, 5 miles.
11. ,, 1 hour................................ ,, 3$\frac{4}{5}$ of 365$\frac{1}{4}$ dys.
12. ,, 2$\frac{7}{10}$ of $\frac{3}{4}$ of 15 solid inches .... ,, 3$\frac{2}{3}$ cubic yd.

SIMPLIFICATION OF FRACTIONS; INCLUDING EXAMPLES IN THE COMPOUND RULES, WHICH INVOLVE FRACTIONS.

81. Under this head are arranged—1st. Exs. involving the expression of frac^{l} quan^{s} in positive terms, and requiring both Addition and Subtraction, as—

I. £$\frac{5}{8}$ + $\frac{3}{4}$ of 21*s*. − $\frac{7}{8}$ of 27*s*.

2ndly. Exs. in Compound Addition, Subtraction, Multiplication, and Division, in which fractional quantities are involved. I give one of each.

II. £3 6*s*. 8$\frac{5}{6}$*d*. + £2 11*s*. 8$\frac{7}{8}$*d*. + £4 13*s*. 9$\frac{3}{8}$*d*.

III. £7 16*s*. 8$\frac{5}{6}$*d*. − £3 11*s*. 10$\frac{7}{8}$*d*.

IV. £8 14*s*. 2$\frac{7}{12}$*d*. × 46$\frac{3}{5}$.

V. £38 3*s*. 4$\frac{3}{8}$*d*. ÷ 13$\frac{7}{8}$.

82. In Ex. I. we may use either of the following methods, viz. reduce all the quantities to the same denomination, and then find the value of their sum as in (62); or, express in positive terms each fraction separately, and then find the value of the whole by Compound Addition and Subtraction.

By the first method, reducing the quantities to the fraction of £1, I have

$$\frac{5}{8}£+\frac{3}{4}\text{ of }21s.-\frac{7}{8}\text{ of }27s.$$

$$=\frac{5}{8}£+\frac{3}{4}\times\frac{21}{20}£-\frac{7}{8}\times\frac{27}{20}£$$

$$=\left(\frac{5}{8}+\frac{63}{80}-\frac{189}{160}\right)£$$

$$=\frac{100+126-189}{160}£ \qquad (\text{since L. C. D.}=160)$$

$$=\frac{226-189}{160}£=\frac{37}{160}£$$

and $\frac{37}{160}£=\frac{37\times\cancel{20}}{\underset{8}{\cancel{160}}}s.=\frac{37}{8}s.=4\frac{5}{8}s.=4s.\ 7\frac{1}{2}d.$

By the second method—

$\frac{5}{8}£$ (64) $=12s.\ 6d.$

also, $\frac{3}{4}$ of $21s.=\frac{63}{4}s.=15\frac{3}{4}s.=15s.\ 9d.$

and $\frac{7}{8}$ of $27s.=\frac{189}{8}s.=23\frac{5}{8}s.=£1\ 3s.\ 7\frac{1}{2}d.$

therefore $\frac{5}{8}£+\frac{3}{4}$ of $21s.-\frac{7}{8}$ of $27s.=12s.\ 6d.+15s.\ 9d.-£1\ 3s.\ 7\frac{1}{2}d.$
$=£1\ 8s.\ 3d.-£1\ 3s.\ 7\frac{1}{2}d.$
$=4s.\ 7\frac{1}{2}d.$

83. Ex. II. $£3\ 6s.\ 8\frac{5}{6}d.+£2\ 11s.\ 8\frac{7}{8}d.+£4\ 13s.\ 9\frac{2}{3}d.$

Commencing with the addition of the fractional parts, I have

$$\frac{5}{6}d.+\frac{7}{8}d.+\frac{2}{3}d.=\frac{20+21+16}{24}d.$$

$$=\frac{57}{24}d.=\frac{19}{8}d.=2\frac{3}{8}d.$$

£	s.	d.
3	6	$8\frac{5}{6}$
2	11	$8\frac{7}{8}$
4	13	$9\frac{2}{3}$
10	12	$3\frac{3}{8}$

and carrying the 2*d.* to the row of pence, I complete the sum as in Compound Addition.

84. Ex. III. $£7\ 16s.\ 8\frac{5}{8}d.-£3\ 11s.\ 10\frac{7}{8}d.$

Commencing with the subtraction of the fractional parts, I observe that $\frac{5}{6} < \frac{7}{8}$; therefore, borrowing 1*d.* from the 8*d.* in the former quantity, I have

$$1\tfrac{5}{6}d. - \tfrac{7}{8}d. = \left(\frac{11}{6} - \frac{7}{8}\right)d. = \frac{44-21}{24}d. = \frac{23}{24}d.$$

$$\begin{array}{rrr} \pounds & s. & d. \\ 7 & 16 & 8\frac{5}{6} \\ 3 & 11 & 10\frac{7}{8} \\ \hline 4 & 4 & 9\frac{23}{24} \\ \hline \end{array}$$

and complete the sum by Comp^d Subtraction.

Ex. IV. £8 14*s.* $2\frac{7}{12}$*d.* × $46\frac{8}{9}$.

Now, since to multiply by $46\frac{8}{9}$ is to repeat the multiplicand 46 times and $\frac{8}{9}$ times, therefore, if I multiply the given comp^d quan^y by 46, and then take $\frac{8}{9}$ of the same, the sum of the two products will be the product required, viz. $46\frac{8}{9}$ times £8 14*s.* $2\frac{7}{12}$*d.* Proceeding as in Comp^d Mult^n, and working in the margin those parts of the mult^n which involve fractions, I have

$$\begin{array}{rrrl} \pounds & s. & d. & \\ 8 & 14 & 2\frac{7}{12} & \\ & & & 9\times5+1=46 \\ \hline 78 & 7 & 11\frac{1}{4} & \\ & & 5 & \\ \hline 391 & 19 & 8\frac{1}{4} & =45 \text{ times} \\ 8 & 14 & 2\frac{7}{12} & = \ 1 \ \ ,, \\ 7 & 14 & 10\frac{8}{27} & = \ \frac{8}{9} \ ,, \\ \hline 408 & 8 & 9\frac{7}{54} & =46\frac{8}{9} \ ,, \\ \hline \end{array}$$

$$\begin{array}{rrrr} & \pounds & s. & d. \\ & 8 & 14 & 2\frac{7}{12} \\ & & & 8 \\ \hline 9) & 69 & 13 & 8\frac{2}{3} \\ \hline & 7 & 14 & 10\frac{8}{27} \\ \hline \end{array}$$

The $\frac{8}{27}d.$ is thus obtained: after dividing the pence by 9, there remain 2*d.*; therefore I have to divide $2\frac{2}{3}d.$ by 9: the quotient

$$= \frac{2\frac{2}{3}d.}{9} = \frac{\frac{8}{3}}{9}d. = \frac{8}{27}d.$$

$$\text{Sum of the fractions} = \left(\frac{1}{4} + \frac{7}{12} + \frac{8}{27}\right)d. = \frac{27+63+32}{108}d.$$

$$= \frac{122}{108}d. = \frac{61}{54}d.$$

$$= 1\tfrac{7}{54}d.$$

Again $46\frac{8}{9} = \frac{422}{9}$; therefore I may multiply the given sum by $\frac{422}{9}$; *i. e.* multiply by 422, and divide by 9. But as this second method involves more labour, and would therefore be not so generally used, it is not worth while to work the Ex. out, especially as the process of Compound Multiplication, where fractions are involved, is sufficiently shown in the former method.

85. Ex. v. £38 3*s.* $4\frac{3}{5}$*d.* ÷ $13\frac{7}{9}$.

There is but one mode of working this Ex., viz. to reduce the divisor to an impr frn—invert it, and proceed as in multiplication.

Now, $13\frac{7}{9}=\frac{124}{9}$; and this, when inverted, becomes $\frac{9}{124}$. Therefore the Ex. becomes £38 3*s.* $4\frac{3}{5}$*d.* × $\frac{9}{124}$; *i. e.* I have to multiply the given sum by 9, and divide by 124. The work will be as follows:—

```
           £   s.  d.
          38   3   4 3/5
                   9            9 × 3/5 d. = 27/5 d. = 5 2/5 d.
     ─────────────────
124)  343  10  5 2/5  (£2
      248
      ───
       95
       20
     ─────
124)  1910  (15s.
      124
      ────
       670
       620
       ───
        50
        12
     ─────
124)   605  (4d.
       496
       ───
       109
```

There now remain 109*d.* and $\frac{2}{5}$*d.*, which have not yet been divided by 124; the quotient =

$$\frac{109\frac{2}{5}d.}{124}=\frac{\frac{547}{5}}{124}d.=\frac{547}{5\times124}d.=\frac{547}{620}d.$$

and the whole quotient is £2 15*s.* $4\frac{547}{620}$*d.*

Exs. 18. Find the value of

1. $\frac{3}{4}$ guinea $+\frac{3}{14}$ of 13*s.* 4*d.*$-\frac{1}{9}$ of 27*s.*
2. $\frac{3}{5}$ ton $+\frac{7}{8}$ lb. $+\frac{5}{12}$ cwt. (in lbs. and oz.)
3. $\frac{3}{5}$ mile$-\frac{4}{9}$ furlong $+\frac{1}{11}$ pole.
4. $1\frac{1}{5}$ acr. $+\frac{4}{7}$ rood$-\frac{3}{8}$ sq. yd.
5. £7 16*s.* $8\frac{1}{6}$*d.* + £138 4*s.* $10\frac{5}{8}$*d.* + £78 6*s.* $3\frac{5}{8}$*d.*
6. 27 yds. 1 ft. $8\frac{3}{5}$ in. + 5 yds. 0 ft. $9\frac{2}{5}$ in.$-$2 ft. $0\frac{3}{8}$ in.
7. 4 tons 3 cwts. $2\frac{5}{9}$ qrs. + 15 cwt. $57\frac{1}{3}$ lbs.$-$3 qrs. 13 lbs. $5\frac{1}{5}$ oz.
8. £463 12*s.* $4\frac{3}{8}$*d.* ∼ £764 19*s.* $3\frac{1}{7}$*d.*
9. £114 8*s.* $0\frac{6}{7}$*d.* × $8\frac{5}{9}$.
10. £15 9*s.* $8\frac{11}{15}$*d.* × $23\frac{5}{7}$.
11. 17 yrs. 3 mo. 2 wks. 3 dys. × $375\frac{8}{11}$.
12. £135 0*s.* $11\frac{3}{16}$*d.* ÷ $12\frac{11}{12}$.
13. 1684 lbs. 10 oz. 17 dwt. $11\frac{1}{3}$ grs. ÷ $47\frac{2}{5}$.
14. £13 15*s.* $9\frac{1}{11}$*d.* × $17\frac{3}{5}$.
15. (£3 8*s.* $7\frac{1}{4}$*d.* + £2 17*s.* $5\frac{1}{3}$*d.*) ÷ $15\frac{1}{3}$.

MISCELLANEOUS EXAMPLES.

86. These, of course, consist of questions involving one or more of the various processes which we have considered; and all that can be done towards guiding the pupil in working such Exs. is, merely to assist him in judging what operations are required in solving any proposed question.

What was said in (60) applies more particularly here, viz. that it is necessary most carefully to note the *signs*, or rather the words which express signs. As an instance of this caution, I will consider the subjoined Ex.; and in order to work it, I shall endeavour to see what operations are intended by the words employed in the question.

Ex. I. Multiply the sum of $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{5}{6}$, and 4, by $7\frac{11}{12}$.

Now *the sum of* means addition, or (+), and *of* means (×); therefore, this Ex. expressed in signs is $\{\frac{2}{3}+(\frac{3}{4}\times\frac{5}{6})+4\}\times 7\frac{11}{12}$. This is now a mere Ex. of simplification as in (60); and the remaining work may be completed by a pupil.

87. Ex. II. How many persons may receive each $3\frac{5}{6}s.$ out of $£13\frac{4}{5}$?

This is only a Reduction sum, involving fractions; and, in plainer language, means—how often are $3\frac{5}{6}s.$ contained in $£13\frac{4}{5}$? or, if I divide $£13\frac{4}{5}$ by $3\frac{5}{6}s.$, what is the quotient?

The quotient required is $\dfrac{£13\frac{4}{5}}{3\frac{5}{6}s.}$

$$=\frac{13\frac{4}{5}\times 20s.}{3\frac{5}{6}s.}=\frac{\dfrac{69}{5}\times\dfrac{20s.}{1}}{\dfrac{23}{6}s.} \qquad \text{(W)}$$

$$=\frac{\overset{3}{\cancel{69}}}{\cancel{5}}\times\frac{\overset{4}{\cancel{20}}}{1}\times\frac{6}{\cancel{23}}=72.$$

By (67) we know that the second fraction in line (W) will be an abstract number; and the result, 72, shows that $3\frac{5}{6}s.$ are contained 72 times in $£13\frac{4}{5}$.

88. Ex. III. Compare $£\frac{3}{7}$, $\frac{3}{8}$ of a guinea, and $\frac{2}{3}$ of $11s.\ 10\frac{1}{4}d.$

We cannot compare quantities without bringing them to some common name; *i. e.* in order to compare fractional quantities, we must reduce them to their L. C. D.

These quantities may therefore either be expressed in positive terms, as pounds, shillings, pence, &c., and then compared; and this is the best method when we wish to know the *difference* in value between the quantities; or they may be all expressed as fractions of one common quantity —say, of £1, or of 1 guinea, and then reduced to L. C. D. I use the latter method, because then I see more clearly the *ratio* between the quantities.

Reducing to fr^{ns} of £1, I have

$$\frac{3}{8}\text{ guinea}=\frac{3}{8}\times\frac{21}{20}£=\frac{63}{160}£.$$

$$\text{Also, }\frac{2}{3}\text{ of }11s.\ 10\frac{1}{4}d.=\frac{2}{3}\times\frac{142\frac{1}{4}}{240}£=\frac{\cancel{2}}{3}\times\frac{569}{\underset{2}{\cancel{4}}\times 240}£=\frac{569}{1440}£.$$

Hence the 3 given quan^s, when expressed in the same den^n, are

$$\frac{3}{7}£,\quad \frac{63}{160}£,\quad \text{and } \frac{569}{1440}£.$$

or, reduced to L. C. D. $\frac{4320}{10080}$, $\frac{3969}{10080}$, $\frac{3983}{10080}$.

or in the ratio of 4320 : 3969 : 3983.

89. I will work in full the following Ex., because upon the method of working it depends the solution of many questions in Arithmetic and Algebra.

Ex IV. If $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$ of a number amount to 36, what is the number?

$$\text{Now, } \frac{1}{3}+\frac{1}{4}+\frac{1}{12}=\frac{4+3+1}{12}=\frac{8}{12}=\frac{2}{3}.$$

And, by the question, this sum = 36;

i. e. $\frac{2}{3}$ of the number = 36;

therefore, dividing both sides by 2,

$\frac{1}{3}$ of the number = 18:

and, multiplying by 3,

$\frac{3}{3}$ of the number, *i. e.* the whole number $=3\times 18=54$.

90. The following is an Ex. which is to be worked upon a principle similar to the last.

Ex. V. If A can do a piece of work in 3 hours, B in 5 hours, and C in 7 hours, in what time can they do it, all working together?

Now, A can do the work in 3 hours;

therefore A can do $\frac{1}{3}$ of the work in 1 hour;

so, B ,, $\frac{1}{5}$,, 1 hour;

and C ,, $\frac{1}{7}$,, 1 hour;

therefore the three, A, B, C, working together, can perform

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7},\text{ or } \frac{35+21+15}{105},\text{ or } \frac{71}{105}\text{ in 1 hour;}$$

therefore, if they do $\frac{71}{105}$ in 1 hour,

they will do $\frac{1}{105}$ in $\frac{1}{71}$ hour, and therefore $\frac{105}{105}$, or the whole in $\frac{105}{71}$ hours;

that is, in $1\frac{34}{71}$ hours.

By observing the frⁿ $\frac{71}{105}$, expressing the amount done in 1 hour, and $\frac{105}{71}$, the number of hours required for the whole work, we find that they are the reverse of one another; and this we might have expected, for the quantity of work done in any given time bears an *inverse* ratio to the amount of time in which it is done (76).

Here I have been finding the time of doing the whole work. I give one more Ex., in which it is required to find how much of a given piece of work can be done in any fixed time.

Ex. vi. A cistern is filled by two spouts in 20 and 24 minutes respectively, and emptied by a tap in 30 minutes; what portion of it will be filled in 15 minutes, when they are all left open together, the influx and efflux being uniform?

Since the 1st tap would fill the whole in 20 min.
therefore it would fill $\frac{1}{20}$th in 1 min.
so also, the 2nd „ $\frac{1}{24}$th in 1 min.

therefore $\frac{1}{20}+\frac{1}{24}$ are poured in by both together in 1 minute; but $\frac{1}{30}$ is discharged in 1 minute by the third tap; therefore, subtracting the quantity discharged from that poured in, we have remaining in the cistern at the end of 1 minute, $\frac{1}{20}+\frac{1}{24}-\frac{1}{30}$;

$$\text{and this} = \frac{6+5-4}{120} = \frac{11-4}{120}$$

$$= \frac{7}{120} \text{ in 1 minute;}$$

therefore at the end of 15 minutes there remains $\cancel{15} \times \frac{7}{\underset{8}{\cancel{120}}}$ or $\frac{7}{8}$; hence in 15 minutes the cistern will be seven-eighths full.

Exs. 19. A.

1. Explain the terms *product*, *dividend*, *quotient*.

2. With 263 for quotient, 368,469 for dividend, and 6 for remainder, what is the divisor?

3. Find the difference between one million and ten, and ten thousand and nine.

4. Write an equation involving the signs of addition, subtraction, multiplication, and division.

5. If a man's yearly income be known, what rule must be used to find his daily income? Ex. If the income be £532 5*s.* 10*d.*, what is that per day?

6. What is the smallest number that can be exactly divided by the nine digits?

7. If light travels 192,000 miles per second, and the sun's light reaches us in 8¼ minutes, how far is the sun from the earth?

8. If 360 degrees be passed over in 365¼ days, how much is that per day?

9. To how many persons may £4 13*s.* 6*d.* each be given, out of a sum of £79 9*s.* 6*d.*?

10. What sum of money will be required to distribute to 10 poor men, 15 women, and 20 children, the respective sums of 2*s.*, 1*s.*, and 6*d.*?

11. Out of an income of 500 guineas, it is desired to lay by £150; what must be saved and spent daily?

12. Compare the product and quotient of 2½ and 1⅞.

13. Give the rules for multiplying and dividing fractions by whole numbers, and illustrate the processes by examples.

14. From unity subtract ½, add ⅓, subtract ¼, and add ⅕; find the ratio of the result to the fraction ⅘ of 1⅜.

15. The sum realized by a bankrupt's estate is £7848, being $\frac{3}{16}$ of his debts, find the amount of the debts, and the dividend paid.

B.

1. How many dollars, each 4*s.* 6*d.*, are contained in 20 moidores?

2. How many times will a wheel, 7 ft. 4 in. in circumference, turn in 3¾ miles?

3. If a railway carriage move 42 feet per second, how many miles is that per hour?

4. Find the value of 1000 oz. of silver plate, at 13*s.* 9*d.* per oz.; and shew how many times more valuable the same weight would be in gold, at £13 8*s.* 1½*d.* per oz.

5. A yard of Cambridge butter weighs 1 lb., what should be the length of one penny-worth, at 17½*d.* per lb.?

6. What sum must be divided among 18 men, and 9 women, so that each man may have £1 3*s.* 6*d.*, and each woman ⅔ of that sum?

7. A cask is required which can be filled by any one of the following measures, taken any number of times exactly, ½ pint, ½ gallon, 3 gals., 5 gals., and 9 gals.; find the smallest cask for this purpose.

8. Find the value of $\frac{7}{12} \sim \frac{9}{10}$ of a lb. Troy.

9. Out of a sum of £18$\frac{9}{10}$, how many persons may receive 2*s.* 7½*d.* each?

10. Shew whether 3 : 19 is greater or less than 2¼ : 14½; and find the change in the last term, that the four may form a proportion.

11. Of a package of cloth, ⅓ is sold at 2*s.* 6*d.* a yard; ¼ at 1*s.* 6*d.* a yard, and the remaining 25 yds. at 1*s.* 10*d.*, gaining 2*d.* per yd.; find the whole gain on the package.

12. Find the product and quotient of $\frac{2}{1\frac{3}{4}}$ and $\frac{1\frac{3}{5}}{4}$.

13. What fraction of £2 15*s.* is £2 14*s.* 9*d.*?

14. If ⅓ of a ship be worth £73 1*s.* 3*d.*, what part of her is worth £250 10*s.*?

15. Find the G. C. M. of 324 and 720, explaining the truth of the process.

C.

1. Express in words (1) the sum, and (2) the difference of 4,000,309 and 70,002.

2. Find in lbs. the value of 3 tons 14 cwt. 57 lbs.—1 ton 17 cwt. 3 qrs.

3. In coining 40,000 penny pieces, each costing ⅞*d.*, how much profit was made?

4. What number of steps 2 ft. 11 in. long, will be taken in walking 45 yards?

5. The cylinder of a railway engine is 18 in. long; how far will the piston travel in 500 revolutions of the driving wheel?

6. In 56 guineas, as many pounds, moidores, and half-crowns, how many groats?

7. If 25 yds. of cloth cost £7 17*s.* 6*d.*, shew without using any proportion statement what is the value of 36 yds.

8. If £1 sterling be worth 26 francs 50 cents, find the number of francs that can be obtained for 1000 guineas.

9. A clock gains 7½ min. per day, find the gain per minute.

10. How many days would it take to count 800 million coins, at the rate of 125 per minute?

11. Find the L. C. M. of 6, 8, 12, 18, 24, 27, explaining the process that you use.

12. Express in the smallest integers the ratio of $\frac{3}{1\frac{1}{3}}$ to $\frac{6\frac{1}{2}}{5}$.

13. After using $\frac{3}{5}$ of a cheese, $\frac{2}{3}$ of the remainder sold for £1 2*s*. $5\frac{1}{2}$*d*.; what was the whole cheese worth?

14. From the sum of $\frac{3}{5}$ of 7*s*., and $\frac{3}{4}$ of half a guinea, take $\frac{2}{5}$ of a guinea, expressing the result as the fraction of a moidore.

15. What number added to $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{2}{7}$ of $\frac{5}{8}$, will make a sum total of 5?

D.

1. Find the value of 3645 × 700705 ÷ 37.

2. How many acres in a field 202 yds. long, and 167 yds. broad?

3. A bar of gold, valued at £3 17*s*. $10\frac{1}{2}$*d*. per oz., is sold for £1429 0*s*. $1\frac{1}{2}$*d*.; what was its weight?

4. A sheet of letter-press contains 24 pages, of which $\frac{2}{3}$ are large type, having 32 lines to the page, and 54 letters in a line; and the remainder is small type, with 45 lines to the page, and 64 letters in a line, how many letters are there in the sheet?

5. Out of a yearly income of £1000, £737 17*s*. 6*d*. is spent for 35 years, how much will have been saved in that time?

6. In a foot-race, where 50 yards start are given to A, the hindermost B gains 5 feet in every 50 yds.; where will the competitors be at the end of a mile?

7. The duty on tea was formerly 2*s*. $2\frac{1}{4}$*d*. per lb., how much must pay the duty, to make 5 millions sterling?

8. How many plots of land, each 50 square perches, can be made out of a square mile?

9. The pendulum of a church-clock vibrates 15 times in 4 minutes; how many vibrations in 24 hours?

10. Define proper, improper, and compound fractions, giving two examples of each.

11. Compare $\frac{11}{15}$, $\frac{2\frac{1}{3}}{7\frac{1}{5}}$, and $\frac{2}{5}$ of $\frac{7}{9}$.

12. A person receives £750 for $\frac{3}{16}$ of his share of a mine worth £10000; what fractional part of the whole was his share?

13. What is the ratio, expressed in integral numbers, between the sum and difference of $17\frac{2}{9}$ and $27\frac{1}{11}$?

14. If I gain $18\frac{3}{5}$*s*. in 15 guineas, how much is that in the pound?

15. Three men, A, B, C, can do a piece of work in 2, $2\frac{1}{2}$, and $3\frac{1}{2}$ hours respectively, how much of the work could be done in 20 minutes by them all working together?

DECIMAL FRACTIONS,

OR

DECIMALS.

91. It is known to all who understand *Numeration*, that where several figures are placed in a horizontal row, so as to form one number, the value of any figure depends upon its distance from the figure nearest to the right, or, as it is commonly called, from the units' place. Thus, if we take the number 6666, we know that the four figures, counting from left to right, have these values respectively—6000, 600, 60, 6; where we observe that the first 6 to the left has 10 times the value of the second—the second, 10 times the value of the third, and so on: or, going from right to left, each figure has one-tenth of the value of the one preceding it; in other words, any figure, when moved from right to left is multiplied by 10 every step, and when moved from left to right is divided by 10 every step. For example, in the number

$$\overset{D}{6}\ \overset{C}{6}\ \overset{B}{6}\ \overset{A}{6},$$

if I move 6 from A to D, or three places to the left, I in reality multiply it by 10 × 10 × 10, or 1000, *i. e.* the 6 becomes 6000. Again, to change a 6 from C to A, or two places to the right, I divide it by 10 × 10, or 100; *i. e.* the figure which before represented 600 now represents 6.

92. Since, then, it has been shewn that successive figures to the right are found by dividing by 10, let this division be continued beyond the units' place; we ought,

therefore, to have, as the value of the first figure to the right of the units' place, $\frac{1}{10}$th of 6, or $\frac{6}{10}$; of the next, $\frac{1}{10}$th of $\frac{6}{10}$, or $\frac{6}{100}$; so, of the next, $\frac{6}{1000}$; of the next, $\frac{6}{10000}$, &c. And when these four additional figures are placed to the right of the units' place, the entire number will be $6666{\cdot}\overset{E}{6}\overset{F}{6}\overset{G}{6}\overset{H}{6}$, where a point has been set after the units' place, to show where the new figures commence. The entire number now consists of 6 thousands, 6 hundreds, 6 tens, 6 units (or 6), 6 tenths, 6 hundredths, 6 thousandths, 6 tenths of thousandths.

Also, observing ${\cdot}\overset{E}{6}\overset{F}{6}\overset{G}{6}\overset{H}{6}$, it may be seen that the same rule holds that was true of the whole numbers, viz. that to move any figure from right to left is to multiply by 10 every step, and from left to right, is to divide by 10. For example: if I change the second 6 from F to H, or move it *two* places to the right, I change it from $\frac{6}{100}$ to $\frac{6}{10000}$, which $= \frac{6}{100} \div 100$; *i. e.* I have divided it by 10 × 10, or by ten *twice*.

Again, if I change the fourth 6 from H to E, or move it *three* places to the left, I change it from $\frac{6}{10000}$ to $\frac{6}{10}$, which $= \frac{6}{10000} \times 1000$; *i. e.* I have multiplied $\overset{H}{6}$ by $10 \times 10 \times 10$, or by 10 three times; hence the same law holds both on the right and left of the point.

DEF. If a number be multiplied by itself any number of times, it is said to be raised to a power. Thus, the multiplication of $2 \times 2 \times 2$ is otherwise expressed by saying that 2 is raised to the power of 3; and a small figure 3 placed to the right of the 2 and above the line (thus, 2^3) indicates or explains how many factors, 2, have been multiplied together. In like manner, 10^4 expresses the multiplication of four factors, each 10, and is called the fourth power of 10.

This small figure is called the *Index*, or *Exponent*.

93. It appears that the figures on the right of the point in reality represent fractions, as $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{1000}$, &c., all of which have as denominators either 10, or powers of 10; hence they are called *Decimal* Fractions, or Decimals. And the point which is placed to separate the whole numbers from the decimals is called the *decimal point*.

Obs. For the future, in speaking of Decimal Fractions, I shall use the single word *Decimals*; and for Vulgar Fractions, the word *Fractions*.

94. We have just seen (91) that in writing down the value which any single figure represents in the whole number 6666, viz. 6000, 600, 60, 6, we place as many ciphers at the right hand of each 6, as will keep it in the place which it had in the number 6666: so also, in the number ·6666, if we wish to write down the value of each one of these four figures separately, we shall have to place as many ciphers to the left of each 6 as will keep it in the place which it had in the number ·6666, or at its proper distance from the *point*. Hence these four figures, when placed singly, would be

·6, ·06, ·006, ·0006, (X)

and, as in whole numbers we might go farther to the *left*, and have as the next figure 60000; so, by going farther to the *right* in the decimals, I should have as the next figure ·00006.

Comparing the four quantities in line (X) with the value which we have shewn to be due to them, viz. $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{1000}$, &c., we have this connection,

$$\frac{6}{10} = \cdot 6 \qquad \frac{6}{100} = \cdot 06 \qquad \frac{6}{1000} = \cdot 006 \qquad \frac{6}{10000} = \cdot 0006$$

where it will be observed, that the number of *ciphers* in the den^r of the left-hand side equals the entire number of figures after the decimal point, whether ciphers or not, on the right-hand side.

95. Taking this fact as proved when there is but one figure in the num^r of the decimal frac^n, I shall now shew that this is true, whatever be the number of figures in the num^r of the frac^n; for example, that $\frac{675}{10000} = \cdot 0675$, where there are four *ciphers* in the left-hand den^r, and four *figures* after the decimal point.

$$\text{For } \frac{675}{10000} = \frac{600}{10000} + \frac{70}{10000} + \frac{5}{10000}$$
$$= \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000};$$

and these, by what has just been shewn, in last Article,

$= \cdot 06 + \cdot 007 + \cdot 0005$

$=$ 6 hundredths $+$ 7 thousandths $+$ 5 tenths of thousandths

$= \cdot 0675$;

i. e. $\frac{675}{10000} = \cdot 0675$, which was the required result.

Hence any frac^n, having as a den^r any power of 10, can be immediately written as a decimal, by writing down the num^r only, and so placing the decimal point that it shall have as many figures on its right hand, as there are ciphers in the den^r of the given frac^n. And if the num^r does not contain as many figures as there are ciphers in the den^r,—that is, as many as it is necessary to have after the point, the amount must be made up by placing as many ciphers between the point and the figures taken from the num^r as shall complete the desired number.

$$\text{Thus, } \frac{3275}{1000} = 3\cdot 275 \qquad \frac{743}{100000} = \cdot 00743.$$

In the latter Ex. I find it necessary to place two ciphers between the point and the 743, to make the number of figures to the right of the point equal to the number of ciphers in the left-hand den^r.

The pupil should now be able to read any decimal either in one sum, or express it in terms of the several parts of which it is composed. Thus, 3·275 may be read in terms of its several den^{ns}, viz. 3, and 2 tenths, 7 hundredths, and 5 thousandths: or 3, and 275 thousandths; and here it will be observed that when a decimal is expressed in one den^n, that den^n will be the lowest contained. Thus in reading ·00743, since the 3 stands for hundredths of thousandths, therefore ·00743 stands for 743 hundredths of thousandths.

96. The position of the decimal point determines the value of every figure both on the right and left of it, that is, both of the whole numbers and the decimals. Therefore, to move the point to the right has the same effect as moving all the figures to the left; and to move the point to the left, is equal to moving the figures to the right.

Now, it has been shewn (92) that to move a figure one place to the left is to multiply it by 10; therefore, if in any number containing a decimal point, I move the point one place to the right, I in reality multiply every figure in the number, and therefore the entire number, by 10: similarly, if I move the point to the left one place, I divide it by 10. Hence, if I wish to multiply a number containing a decimal point by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the right; and if I wish to divide the number by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the left.

Ex. 3275·468. (Y)

Moving the point two places to the right, the number becomes 327546·8; and it will be found that any figure has now 100 times the value that it had before. The 5 formerly stood for *5 ones*, or *5*, but now for *5 hundreds*, or 500; the 6 for 6 hundredths, or $\frac{6}{100}$, but now for 6, or $\frac{6}{100} \times 100$; that is, each figure has been multiplied by 100, merely by moving the point two places to the right.

Next, let the point be moved three places to the left, and the number becomes 3·275468; and we then see that the 7, which in (Y) was 70, is now only $\frac{7}{100}$, or $\frac{70}{1000}$; that is, has one-thousandth of its former value: so, also, the 3, which before represented 3000, now represents only 3; hence it appears that the entire number has been divided by 1000, merely by moving the point three places to the left.

Exs. 20.

I. Multiply 8·034 by 10, 10,000, 100,000, and 1,000,000 successively.
II. Divide 175·04 by 100, 100,000, 1,000,000, and 10 successively.
III. Multiply ·005 by 10^3, 10^5; and divide it by 10, 10^2, 10^4.

CONVERSION OF VULGAR FRACTIONS INTO DECIMALS.

97. In the use of Decimals we shall find it necessary to know how to convert fractions into decimals. All examples of this kind will in reality be particular examples of Division of Decimals; but as the simpler ones can be worked mentally, or by Short Division, it is well to work them independently of the general rule of division. And it will be seen that there are two classes of fractions, one containing those which can be exactly expressed as decimals, and the other, such as cannot. We shall presently shew the ground upon which this variety rests; and shall first treat of the former class, which includes all fractions which, *when in their lowest terms*, have, as den^{rs}, numbers which contain the figures 2 and 5 as their only factors. These may be called *convertible* fractions.

98. We have seen that a frac^{n} whose den^{r} is a power of 10, can immediately, by inspection, be converted into a decimal. And if the den^{r} be any other than a power of 10, it must be got rid of by dividing the num^{r} by the den^{r}.

Any den^{r} which contains an equal number of both *twos* and *fives* will evidently contain as many factors 10, as 2 and 5, and therefore be of the form 10, 100, 1000, &c. Take for example as a den^{r} $2 \times 2 \times 5 \times 5 = 10 \times 10 = 100$: and when a fr^{n} with such den^{r} is converted into a decimal, there will plainly be as many decimal places as there are *twos* and *fives*.

Exs. 21. Express as decimal fractions

I. $\frac{3}{10}$, $\frac{11}{1000}$, $\frac{19}{100}$, $\frac{15}{100,000}$, $\frac{1001}{10}$

99. Next, let the den^{r} contain factors 2 only, or be a power of 2, as $\frac{3}{8}$ or $\frac{3}{2^3}$.

Now, since the 8 cannot be contained in the 3 units, I must therefore bring these units into tenths, and say, 8 in 30 tenths, which gives 3 *tenths* as quotient, and 6 tenths over; and since 6 tenths = 60 hundredths, 8 in 60 hundredths gives 7 *hundredths*, and 4 hundredths over, or 40 thousandths; 8 in 40 thousands gives 5 *thousandths;* and there being no remainder, the division terminates: and collecting the three quotients, viz. 3 tenths, 7 hundredths, and 5 thousandths, we plainly have $\frac{3}{8} = \cdot375$.

It will be seen that like as there are *three* factors 2 in the den^{r} 8, so there are *three* places in the resulting decimal. The reason of this may be seen by representing the process thus, where we multiply num^{r} and den^{r} by 1000,

$$\frac{3}{8} = \frac{3}{2 \times 2 \times 2} \times \frac{10 \times 10 \times 10}{1000} = \frac{3 \times 5 \times 5 \times 5}{1000} = \frac{375}{1000} = \cdot375.$$

It is here plain that the three factors 2 could not have disappeared in the cancelling, (i.e. in the division by the den^{r} 8,) had there not been also three factors 10 in the num^{r}, and therefore three in the den^{r}, making 1000, and giving three dec^{l} places in the result.

In Simple Short Division, as soon as I know where to put the *first* quotient, I can write down the others in order; so here, as soon as I find that the first quotient consists of tenths, I can place the other quotients without further consideration. But in determining this *first* quotient there is a liability to error; for if a pupil takes the fr^{n} $\frac{3}{80}$, and begins to divide by 80, he may incautiously cut off the ciphers from divisor and dividend, as in Simple Division: but, by placing the fr^{n} thus

$$\begin{array}{r|l} 80 & \underline{\;3\cdot0000} \\ & \;\;\cdot0375 \end{array}$$

it will be found that if I cut off a cipher at the end of the dividend, I remove a figure which has no value; whereas, to take 0 from the 80 diminishes it tenfold: I must therefore say, 80 in 3 units gives as quotient 0 units; so also, since 3 units = 30 tenths, 80 in 30 tenths gives 0 tenths. Again, 30 tenths = 300 hundredths, and 80 in 300 hundredths gives 3 hundredths; hence 0 must be put as quotient under the tenths, and 3 under the hundredths; the remaining quotients will then fall into their proper places by mere Simple Division.

But in examples like this, where the den^{r} contains a power of 10, it is best to divide by the other factors first, and then divide by that power of 10. In the Ex. $\frac{3}{80}$, we should divide both num^{r} and den^{r} by 8, and then divide by 10, by inspection. Thus

$$\frac{3}{80}=\frac{\cdot375}{10}=\cdot0375$$

So, also, $\frac{3}{8000}=\frac{\cdot375}{1000}=\cdot000375$ by (96)

100. If the den^r^ of the fr^n^ be a composite number larger than 12, we may perform the division by breaking the den^r^ into factors, and dividing by them successively. For example

$$\frac{5}{32}=\frac{1\cdot25}{8}=\cdot15625, \qquad \text{(A)}$$

where the second fraction in (A) was obtained by dividing num^r^ and den^r^ by 4.

A fraction whose den^r^ has *fives* only as its factors, must be reduced to a decimal by a process precisely similar to the one exhibited in (98) and (99).

Ex. $\frac{3}{25}$. Here there are two factors 5 in the den^r^; therefore the div^n^ will terminate when I have used 2 ciphers in the num^r^.

$$\text{Thus, } \frac{3}{25}=\frac{\cdot6}{5}=\cdot12$$

$$\text{So, also, } \frac{3}{12500}=\frac{\cdot6}{2500}=\frac{\cdot12}{500}=\frac{\cdot024}{100} \qquad \text{(B)}$$

$$=\cdot00024 \qquad \text{by (96)}$$

where the three factors 5 in 125, (*i. e.* in that part of the den^r^ which is being got rid of,) produce three figures in the num^r^ of the last fraction in (B).

Exs. 21. Express as decimal fractions

II. $\frac{3}{5}$, $\frac{7}{8}$, $\frac{1}{2}$, $\frac{9}{4}$, $\frac{11}{8}$, $\frac{17}{16}$, $\frac{13}{25}$, $\frac{3}{500}$

III. $\frac{7}{125}$, $5\frac{3}{64}$, $11\frac{13}{625}$, $8\frac{1}{5}$ of $11\frac{3}{8}$, $7\frac{1}{80}$ of $5\frac{1}{16}$

The following equalities are worth remembering.

$\frac{1}{8}=\cdot125$ $\quad \frac{1}{4}=\cdot25$

$\frac{3}{8}=\cdot375$

$\frac{1}{2}=\cdot5$

$\frac{5}{8}=\cdot625$

$\frac{7}{8}=\cdot875$ $\quad \frac{3}{4}=\cdot75$

Their utility appears as follows:—if in converting a fraction to a decimal I have a diviser 8, and I come to the last figure in the dividend, one of the above equalities will enable me to write down the remaining quotients immediately.

Thus, if I have to reduce $\frac{75}{8}$ to a decimal, I find that after one quotient 9 has been obtained, 3 is my last dividend; hence, since $3 \div 8 = {\cdot}375$, I can write $\frac{75}{8} = 9{\cdot}375$, without *performing* the division for the last three quotients. Similarly, $\frac{11{\cdot}7}{8} = 1{\cdot}4625$, where the last three figures were obtained, as before, by remembering the value of $5 \div 8$. So also, $\frac{75}{4} = 18{\cdot}75$,—the last two quotients being written without dividing.

101. Coming now to what we may call the *inconvertible* fractions, *i. e.* those which, when reduced to lowest terms, contain other factors than 2 and 5 in the den^r^, as, for instance, 3, 7, 9, &c. we see at once, that the power of 10 which reduces the num^r^ to tenths, hundredths, &c. is not divisible by these factors; therefore they cannot be cancelled out, as the factors 2 and 5 were; hence the division will not terminate. But since the successive quotients diminish tenfold in value every step, so that the eighth to the right of the point represents hundredths of millionths, we generally pursue the division till we obtain 7 figures in the decimal, and then consider the remaining quotients as too small in value to be much appreciated.

Moreover, whatever be the factor which cannot be cancelled out of the den^r^, we know that in dividing by any divisor, the remainder must always be at least one less than the divisor: therefore if any divisor, as 7, be the factor which causes the division not to terminate, there can be but six remainders; and hence, when 7 divisions have been performed, one of these six remainders must come over again, or recur:

and if the remainder, which, of course, influences the dividend, recurs, the quotients will also recur, and we shall then have the same set of figures *recurring*, as it is termed.

Ex. To convert $\frac{6}{7}$ into a decimal.

Working mentally, as in (99), I have

$$\tfrac{6}{7} = \cdot857142857142, \text{ \&c.} \qquad \text{(C)}$$

where it appears that the figures 857142 will recur, to whatever length the division may be carried: hence such decimals are called *Recurring Decimals*, and the set of quotients which recurs is called a *Period*.* If the period consist of but one figure, that single figure is written with a dot (·) over it in place of the whole decimal; but if there be more figures than one, the period is written with a dot over the first and last of its figures.

$$\text{Thus, } \frac{6}{7} = \cdot857142857142\ldots\ldots \text{ or } \cdot\dot{8}5714\dot{2}$$

$$\frac{1}{3} = \cdot333\ldots\ldots \text{ or } = \cdot\dot{3}$$

$$\frac{1}{9} = \cdot1111\ldots\ldots \text{ or } \cdot\dot{1}$$

$$\frac{3}{22} = \frac{1\cdot5}{11} = \cdot13636\ldots\ldots \text{ or } \cdot1\dot{3}\dot{6}.$$

OBS. The places which are occupied by the figures to the right of the decimal point are often called *decimal places*. Thus, in the Ex. $\frac{6}{7}$, we say that the division was carried to six places before the figures began to recur.—(See Appendix, Art. Circulating Decimals).

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* Observing the line (C) it may be noticed that whenever a fr^{n} has 7 for its den^{r}, the recurring decimal obtained will consist of the figures 857142; but this period will not always begin with the figure 8, but with some other of the above 6 figures; thus $\frac{3}{7} = \cdot428571$ &c., where it will be observed that the period is the same as in (C), except that we begin with the 4; and if these figures are retained in the memory, the whole may be written down at once, as soon as the first figure has been obtained by division.

If, however, when the den^{r} is 7, the num^{r} be already a recurring decimal, the above observation will not hold, because the additional figures at the end of the num^{r} are not ciphers; as in the above fr^{ns} $\frac{6}{7}$ and $\frac{3}{7}$.

Exs. 22. Express as decimal fractions

I. $\frac{1}{6}$, $\frac{1}{7}$, $\frac{3}{7}$, $\frac{2}{11}$, $\frac{5}{13}$, $\frac{7}{9}$, $\frac{1}{3}$ of $1\frac{2}{3}$

II. $\frac{1}{6}$, $\frac{5}{9}$, $\frac{11}{7}$, $\frac{13}{28}$, $\frac{17}{65}$, $\frac{4}{27}$, $\frac{3}{56}$, $\frac{2}{3}$ of $1\frac{1}{13}$.

III. $8\frac{5}{13}$, $17\frac{4}{33}$, $375\frac{7}{55}$, $\frac{1915}{990}$

TO CONVERT TERMINATING DECIMALS INTO VULGAR FRACTIONS.

102. It will be easily seen that a number expressed as a decimal can be immediately exhibited as a vulgar fraction. For, since it has been shewn (95) that $\frac{675}{10000} = \cdot 0675$, therefore we can reverse the process, and say that ·0675, when converted into a frⁿ, becomes $\frac{675}{10000}$; and the correctness of this conversion may be proved thus:

$$\cdot 0675 = \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= \text{(when reduced to L. C. D.)}\ \frac{600 + 70 + 5}{10000}$$

$$= \frac{675}{10000}. \qquad \text{(D)}$$

$$\text{So also, } 3\cdot 0275 = 3 + \frac{0}{10} + \frac{2}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= \frac{30000 + 200 + 70 + 5}{10000} = \frac{30275}{10000}, \qquad \text{(E)}$$

a fraction greater than 1, as might have been foreseen, because 3·0275 is partly a whole number and partly a decimal.

Def. In a whole or mixed number, the part which is not fractional is called *Integral*, or an *Integer*.

103. In (D) we observe that the numʳ of the vulgar frⁿ into which we have changed the decimal, before it is reduced

to lowest terms, consists of the figures in the given decimal, excluding ciphers not preceded by other figures; and the den^r is 1, followed by as many ciphers as there are decimal places.

But in (E), where a cipher occurs in the given mixed decimal 3·0275, it remains in the num^r of the equivalent improper fr^n; and the den^r is formed as before.

The above fractions may, of course, be reduced to lowest terms; but I have left them in their present shape, in order to shew the connection between the given decimal and the fractional form into which it could be converted by inspection. Reducing them, and writing the Exs. as they should be exhibited in common use, we have

$$\cdot0675 = \tfrac{675}{10000} = \tfrac{27}{400}; \text{ and } 3\cdot0275 = \tfrac{30275}{10000} = \tfrac{1211}{400} = 3\tfrac{11}{400}.$$

If the given number be partly a whole number and partly a decimal, we may leave the integral part unaltered, and then the resulting fr^n will appear at once as a mixed number.

$$\text{Thus, } 76\cdot0725 = 76\tfrac{725}{10000} = 76\tfrac{145}{2000} = 76\tfrac{29}{400}.$$

Exs. 23. Convert into vulgar fractions, in their lowest terms,

I. ·05, 10·73, 115·008, ·0001, 1·005, ·343.

II. 12·01, 10·008, ·00725, 135·55, ·10505, 9·99.

TO CONVERT RECURRING DECIMALS INTO VULGAR FRACTIONS.

104. Though at first it may be thought that a *non-terminating* decimal cannot be accurately represented by a fr^n; yet, since every recurring decimal is formed from one of that class of fractions which was discussed in (101), therefore every such decimal can of course be made to resume the shape from which it was derived: and the accuracy which

was lost in the change from a fraction to a decimal is restored by this reconversion to the original fraction.

105. Circulating Decimals are of two kinds; one in which the whole of the figures repeat, as

$$\cdot363636\ldots\ldots \text{or } \cdot\dot{3}\dot{6};$$

and the other, in which some of the figures to the right of the point are not repeated. These figures, of course, always stand to the left of the circulating part, because when a decimal once begins to circulate, it continues to do so. An Ex. of this kind is

$$\cdot754365365\ldots\ldots \text{or } \cdot754\dot{3}6\dot{5}.$$

The former kind is called a *Pure*, and the latter a *Mixed* Circulating Decimal.

106. The best method of working Exs. under this case will not be intelligible without an acquaintance with one or two facts in algebra. A letter of the alphabet may be used to represent a quantity, the value of which we have to find, as in (73). Thus, I may let F stand for or represent the fraction which is equivalent to any circulating decimal; and it is for me to ascertain the value of this F in any particular example.

Also, a number placed just before this F, and with no sign connecting F and the number, is a multiplier of the F: thus, 10F, $\frac{2}{3}$F, mean ten times F, $\frac{2}{3}$ times F, or $\frac{2}{3}$ of F; and they have the same value as though they were written $10 \times F$, $\frac{2}{3} \times F$. F repeated once is not written 1F, as we should write 1s., but only F. These multipliers 10, $\frac{2}{3}$, and 1, are called *coefficients*. Moreover, since in the quantity 10F, F is the den^{n}, and the 10 tells us how many times F is taken, just as the 7 in 7s. tells us how many shillings are taken, therefore I can add to this 10F, or subtract from it, any number of quantities of the same denⁿ, i. e. any number of times F.

Thus, $10F+3F=13F$ $\qquad 10F-3F=7F$

and $100F+F=101F$ $\qquad 100F-F=99F$.

107. I now proceed to find the frⁿ which is equivalent to any circulating decimal, as

Ex. I. $\cdot\dot{3}\dot{6}$ or $\cdot363636\ldots$

$$\text{Let } F=\cdot363636\ldots \qquad (G)$$

Now if I multiply both sides of this equation by the same number, the equality will not be disturbed. My object is to move the point two places to the right in the decimal ·363636……, so that it shall become 36·3636……This is done by multiplying both sides by 100; the left-hand side therefore becomes $100\times F$, or $100F$; hence we have

$$100F=36\cdot3636\ldots\ldots \qquad (H)$$

$$\text{also } F=\cdot363636\ldots\ldots \qquad (G)$$

Subtracting (G) from (H), we observe that on the left-hand side, F or 1F taken from 100F leaves 99F as in (106); and on the right-hand side, since both decimal parts commence alike and go on for ever, their difference is 0; and after subⁿ there remains only the integral number 36; hence we have

$$99F=36,$$

and dividing both sides by 99, we find

$$F=\frac{36}{99}=\frac{4}{11}. \qquad (I)$$

By observing (I) we learn that the fraction which represents the value of the circulating decimal $\cdot\dot{3}\dot{6}$ has for its numʳ the figures in the period, viz. 36, and for the denʳ, as many figures 9 as there are figures in that period.

In accordance with this rule we may therefore, by inspection, find the value of any circulating decimal which contains only such figures as do repeat.

$$\text{Thus, } \cdot\dot{6}=\frac{6}{9}=\frac{2}{3}$$

$$\text{and } \cdot\dot{1}4\dot{7}=\frac{147}{999}=\frac{49}{333};$$

but every pupil ought to work some Exs. out fully as in the beginning of this article.

108. Again, noticing (H) and (G) we see that the figures to the right of the decimal point are the same in both lines; and this result was produced by so multiplying both sides of the equation (G), that the point might be made to pass over one period; in this case it had to be moved two places to the right; that is, I had to multiply by 10 × 10, or 100. If the period had contained four figures, I should have multiplied by 10^4, or 10000. I will take as an Example

Ex. II. To express $\cdot\dot{2}21\dot{6}$ as a fraction.

$$\text{Let } F=\cdot22162216\ldots\ldots \quad (K)$$

$$\text{herefore, } 10000F=2216\cdot2216\ldots\ldots \quad (L)$$

Subtracting (K) from (L), we have

$$9999F=2216;$$

$$\text{therefore, } F=\frac{2216}{9999}:$$

and it is plain that this result might have been written down at once by taking as num^{r} of the required fr^{n} the period 2216, and as den^{r} a number consisting of as many figures 9 as there are figures in this period.

109. We now proceed to convert a mixed circulating decimal into a fraction. The nature of the process is the

same as in the last article; but one step more is required, the reason of which will appear in the operation.

Ex. III. To convert $\cdot 32\dot{7}1\dot{5}$ into a fraction.

$$\text{Let } F = \cdot 32715715\ldots\ldots \quad (M)$$

Then, as before, (107), carrying the decimal point over the period, or five places to the right—*i. e.* multiplying both sides by 100000, I have

$$100000F = 32715\cdot 715\ldots\ldots \quad (N)$$

It now appears that I cannot subtract (M) from (N), so as to get rid of the decimal parts, because the quantities to the right of the point are not the same: but I see that I can very readily obtain another equation which shall have the decimal part the same as in (N); for if in (M) I multiply both sides by 100, *i. e.* move the point two places to the right, or over the non-recurring part alone, I shall have

$$100F = 32\cdot 715715\ldots\ldots \quad (O)$$

therefore, subtracting (O) from (N), I have

$$[100000F - 100F = 32715\cdot 715\ldots - 32\cdot 715715\ldots] \quad (P)$$

$$\text{or } 99900F = 32715 - 32;$$

$$\text{therefore, } F = \frac{32715-32}{99900} \qquad (Q)$$

$$= \frac{32683}{99900}.$$

110. In the line (P) I learn that the larger coefficient of F has as many ciphers as there are figures from the point to the end of the first period; and the smaller coefficient has 2 ciphers, *i. e.* as many as there are figures in the non-recurring part: also, when the subtraction is *performed* in the next line, the coefficient of F has as many ciphers as there are figures in the non-recurring part: and the remaining figures to the left are all nines, and as many in number as there are figures

in the period. Hence, observing the right-hand side of (Q), I learn that in converting a mixed circulating decimal into a fraction I obtain as a num^r, "the figures of the given decimal to the end of the first period, *minus* the figures in the non-recurring part;" and as den^r, "as many figures 9 as there are figures in the period, followed by as many ciphers as there are decimal places in the non-recurring part." If there be an integer in the given recurring decimal, I may omit it while finding the value of the recurring part; and afterwards by inserting it I shall have the resulting fraction a mixed number; or I may retain it throughout, as in Ex. V. below, and the result will be an improper fraction.

I will write down one more Ex. of each kind of circulating decimals, in the form in which they ought to be worked.

Ex. IV. Find the fraction equivalent to $\cdot\dot{6}3\dot{9}$.

$$\text{Let } F = \cdot 639639639\ldots\ldots \quad \text{(I.)}$$

multiplying by 1000,

$$1000F = 639\cdot 639639\ldots\ldots \quad \text{(II.)}$$

Subtracting (I.) from (II.),

$$999F = 639,$$

$$\text{or } F = \frac{639}{999} = \frac{213}{333} = \frac{71}{111}.$$

Ex. V. Find the fraction equivalent to $2\cdot 0\dot{3}4\dot{5}$.

$$\text{Let } F = 2\cdot 0345345\ldots\ldots \quad \text{(III.)}$$

multiplying by 10000,

$$10000F = 20345\cdot 345\ldots\ldots \quad \text{(IV.)}$$

Again, multiplying (III.) by 10, $10F = 20\cdot 345345\ldots\ldots$ (V.)
and subtracting (V.) from (IV.)

$$9990F = 20345 - 20$$

$$\text{therefore, } F = \frac{20325}{9990} = \frac{6775}{3330} = \frac{1355}{666}$$

$$= 2\tfrac{23}{666};$$

or, omitting the integral part, as described above, I have

$$\text{the fractional part} = \frac{345}{9990} = \frac{69}{1998} = \frac{23}{666},$$

therefore, the entire quantity $= 2\frac{23}{666}$, as before; and this is the better method.

Exs. 24. Convert into vulgar fractions, in their lowest terms,

I. $\cdot\dot{1}$, $\cdot\dot{2}7\dot{3}$, $\cdot\dot{0}\dot{9}$, $\cdot\dot{0}10\dot{1}$, $15\cdot\dot{0}7\dot{5}$, $\cdot\dot{1}4285\dot{7}$.

II. $\cdot1\dot{6}$, $\cdot0\dot{2}\dot{7}$, $130\cdot\dot{2}8571\dot{4}$, $\cdot000\dot{1}\dot{9}$, $\cdot14285\dot{7}$, $35\cdot\dot{0}0\dot{9}$.

III. $\cdot08451\dot{6}$, $60\cdot0101\dot{4}$, $100\cdot0\dot{3}\dot{6}$, $\cdot16\dot{7}843\dot{2}$, $35\cdot00\dot{9}$, $3500\cdot\dot{9}$.

ADDITION.

111. It has been stated in Part I. that the four elementary processes of Addition, Subtraction, &c. are applicable to quantities involving decimals; we have now to shew that the rules there given for their working are true.

Since we already know that quantities cannot be added together unless they be of the same den$^{\text{r}}$, so in the addition of decimals, we must take care to add tenths to tenths, hundredths to hundredths, and so on. And this will be readily done, if we so place all the numbers under one another, that the decimal points may be in a vertical row. By observing the Ex. worked below, we notice that the tenths are all placed under one another, as are also the hundredths, thousandths, &c.

$$\text{Ex. I.}\quad \begin{array}{r@{\cdot}l} 732 & 416 \\ & 084 \\ & 000007 \\ 93 & 268 \\ 2708 & 4153 \\ \hline 3534 & 183307 \\ \hline\hline \end{array}$$

Adding up the third row to the right of the point, which consists of thousandths, I find that it amounts to 23 thousandths, or $\frac{23}{1000}$, which $= \frac{20}{1000} + \frac{3}{1000} = \frac{2}{100} + \frac{3}{1000}$; I therefore put down the 3 thousandths, and carry the 2 hundredths to the next column, which consists of hundredths; and so step is just such as would be performed in Simple Addition, it is plain that all the rest of the work may be performed by that Rule.

112. But if in the quantities to be added there are circulating decimals, we may either, 1st, convert into a fr^{a} each circulating decimal, and having found the sum of all these fractions, reduce the result to a circulating decimal: or, 2ndly, (and this method is the better,) write down the recurring decimals at length, to as many places as will include twice the longest period; observe where there are two vertical columns alike, though not necessarily close to each other; commence the addition three or four places to the right of the second of these two similar columns, and complete the addition, as before: the sum found will be seen to be carried far enough to enable a pupil to detect the period in the answer.

Ex. II. Find the sum of $\cdot\dot{7}1428\dot{5} + \cdot 9\dot{2}8571\dot{4} + 20\cdot 0\dot{9}2\dot{5} + 5\cdot 4\dot{0}4761\dot{9}$.

Writing these at length, and working according to the above directions, I have

$$\begin{array}{r}
\cdot 7\overset{A}{1}428571\overset{B}{1}4285\ldots\ldots \\
\cdot 9285714285714\ldots\ldots \\
20\cdot 0925925925925\ldots\ldots \\
5\cdot 4047619047619\ldots\ldots \\
\hline
27\cdot 14021164020 \\
\hline\hline
\end{array}$$

Here I observe that the first two similar vertical columns are the second and the eighth, marked (A) and (B). Commencing the addition at the third column beyond (B), I find that I have figures enough in the result to shew that the sum of the given circulating decimals is $27\cdot 1\dot{4}0211\dot{6}$.

Exs. 25. Find the value of

1. $18\cdot 325 + \cdot 0007 + 70\cdot 1 + 35\dot{8} + 3\cdot 04705 + 1000\cdot 06$.
2. $347\cdot 859 + \cdot 010101 + 639 + 2\cdot 573 + 11\cdot 01115 + \cdot 32784$.
3. $1\cdot 000009 + 45 + 3845\cdot 1 + 75\cdot 6832 + 10\cdot 01 + \cdot 04311$.
4. $7600 + 3\cdot 1009 + 473\cdot 842691 + \cdot 07 + \cdot 00001 + 1\cdot 1$.
5. $3458 + 5\cdot 143\dot{2}8\dot{5} + \cdot\dot{0}7\dot{5} + 145\cdot\dot{2}\dot{7} + \cdot\dot{8}7516\dot{9}$.
6. $\cdot\dot{3} + \cdot\dot{0}\dot{3} + 145\cdot 27\dot{3}4\dot{5} + 3\cdot\dot{0}0\dot{9} + 6\cdot\dot{1}4285\dot{7}$.
7. $\cdot\dot{6}8\dot{4} + 1\cdot\dot{5}8764\dot{9} + 3\cdot\dot{8}4100\dot{7} + \cdot\dot{0}\dot{7}$.
8. $1\cdot\dot{6} + 19\cdot 3485\dot{4} + \dot{0}\cdot 2\dot{7} + 5\cdot 34785\dot{6} + 111\cdot 1 + \cdot 0\dot{3}\dot{5}$.

SUBTRACTION.

113. As Addition of Decimals was shewn to be only an extension of Simple Addition, so Subtraction of Decimals is of the same nature with Simple Subtraction; but we must take care how we subtract, when the lower line contains more decimal places than the upper.

Ex. I. Find the value of $18{\cdot}0426-2{\cdot}005417$.

Placing the two quantities one under the other, so that the points are in a vertical row, we shall, as in Addition, have tenths under tenths, &c.

As there are no figures in the 5th and 6th places in the upper line, I may place ciphers there; and since there are no figures to the right of these ciphers, the value of the decimal will plainly remain unaltered.

$$\begin{array}{rcr} \text{A} & & \\ 18{\cdot}042600 & \qquad & 600 \\ 2{\cdot}005417 & & 417 \\ \hline 16{\cdot}037183 & & 183 \\ \hline\hline \end{array}$$

If we now take the last three figures in each row as whole numbers, so as to form a Simple Subtraction Sum, we find the difference to be 183: so also, since the 600 and 417 in (A) both represent millionths, their difference is 183 millionths: similarly the 42 and 5 are both thousandths, and their difference = 37 thousandths: hence we see that if the subtraction be performed in (A) as in a common Ex. in Subtraction, the result will be correct.

114. If an Ex. in Subtraction contain circulating decimals, we must, as in Addition, write down the periods sufficient to contain twice the longest period, and commence the subtraction about five places beyond the latter one of the two similar rows; the remainder which will recur will be found to contain figures enough to enable us to detect the period.

Ex. II. Find the value of $24{\cdot}\dot{4}0\dot{7}-5{\cdot}98\dot{0}7697\dot{3}$.

Here, writing five of the periods in the upper line, and two in the lower line, we have

$$\begin{array}{r}
24{\cdot}40\overset{A}{7}40740\overset{B}{7}407407\ldots \\
5{\cdot}98076978076973\ldots \\
\hline
18{\cdot}42663767766377 \\
\hline\hline
\end{array}$$

The columns marked (A) and (B) are the first pair which are similar; I therefore commence the subtraction at the fifth row beyond (B): and the remainder, which contains the circulating decimal, is evidently 18·42$\dot{6}$6376$\dot{7}$.

By observing the Ex. in Addition of circulating decimals, I find that the figures which should form the second period are somewhat below the value of those in the first period; but by commencing with the addition a little more to the right, I should have figures to *carry*, as it is called, which would have made the second period correct as well as the first. So in the Ex. just worked in Subtraction, the figures which should form the second period are too large; but by commencing the Subtraction a *few* more places to the right, the process of *borrowing* would make the figures of the second period quite correct.

Exs. 26. Find the value of

I. 3·5—·075; 175·8—1·0024; ·715—·70451.

II. 1·325—·4736; 10·01—9·875; 1·$\dot{3}\dot{6}$—·75; ·$\dot{3}$—·$\dot{1}$.

III. ·$\dot{3}$—·07; 3·0758—1·$\dot{2}7\dot{3}$; 13·4$\dot{0}8\dot{9}$—6·85$\dot{3}7492\dot{1}$.

MULTIPLICATION.

115. The rule for the multiplication of decimals is to be found by multiplying the vulgar fractions equivalent to any given decimals, and then observing the nature of the product so obtained.

Ex. I. To find the product of 3·275 and 18·03.

We have $3{\cdot}275 \times 18{\cdot}03 = \dfrac{3275}{1000} \times \dfrac{1803}{100}$

$$= \frac{3275 \times 1803}{100000} = \frac{5904825}{100000} \qquad \text{(I.)}$$

$$= 59{\cdot}04825.$$

From (R) we learn that the product of the given decimals is a decimal fraction which has for its numerator the product of the given numbers treated as integers; and for its denominator, 1, followed by as many ciphers as there are decimal places in *both* the multiplier and multiplicand; and such a fraction is of course immediately (95) convertible into a decimal, which shall have as many decimal places as this fraction has ciphers; and this conversion is performed in the next line. Hence, as this Ex. differs not from any other which contains only terminating decimals, we conclude that the product of two such decimals is found by multiplying them together as in whole numbers, and pointing off in the product as many places as are found in both multiplier and multiplicand.

116. Also, if it be required to multiply together more than two numbers involving decimals, it will be found that the same rule must be observed. For suppose that four numbers were given, whose product was required: the product of the first pair might be found, as above; then this product and the third number might be multiplied: and, lastly, this second product and the fourth number: and so on for any number of quantities; hence we may find the product of any number of decimal quantities, as in the previous Example.

Ex. II. Find the value of ·0095 × 2·07 × 7·06 × ·0081.

$$\text{The product} = \frac{95}{10000} \times \frac{207}{100} \times \frac{706}{100} \times \frac{81}{10000}$$

$$= \frac{95 \times 2\text{[illegible]}7 \times 706 \times 81}{1\text{[illegible]}000} = \frac{1124562690}{1000\,000\,000\,000}$$

$$= ·001124562690$$

[illegible] laces, viz. 12, = sum of the numbers of [illegible] s.

[illegible] ave [illegible] illustrating the truth of [illegible] may [illegible] imilar Exs. as in Simple [illegible] d po [illegible] y decimal places in the [illegible] the nu [illegible] ltiplied together.

[illegible] been [illegible] rked as follows;

$$\begin{array}{r}
3{\cdot}275 \\
18{\cdot}03 \\
\hline
9825 \\
262000 \\
3275 \\
\hline
59{\cdot}04825 \\
\hline\hline
\end{array}$$

Much instruction may also be derived from working an Ex. in the following manner:

Ex. III. To find the product of 271·405 and 93·6854.

$$\begin{array}{r@{\cdot}l}
271 & 405 \\
93 & 6854 \\
\hline
814 & 215 \\
162 & 8430 \\
21 & 71240 \\
1 & 357025 \\
 & 1085620 \\
24426 & 45 \\
\hline
25426 & 6859870 \\
\hline\hline
\end{array}$$

Placing the multiplier under the multiplicand in any position whatever, I commence multiplying by the figure in the units' place, viz. 3. Since then I am now merely repeating every figure in the multiplicand 3 times, therefore every figure when multiplied will give a product of the same *kind* as itself, and that product will occupy the same place with respect to the point as it did in the mult$^{\text{d}}$; hence the product will clearly be 814·215.

If, now, I multiply by the 6, which is in the *tenths'* place, the product will be ten times less than it would have been, had the 6 been in the *units'* place; hence I place every figure in this product one place farther to the *right* than in the previous line; so, also, the product by the 8 will be *two* places to the right, by the 5, will be *three* places, &c.

Again, since the 9 in the multiplier represents 90, I place the product obtained by multiplying by the 9 one place more to the *left* than the first product; and if there were any figures in the places for hundreds, thousands, &c. of the multiplier, I should place the corresponding products two, three, &c. places to the left. The whole of the work will now be intelligible.

The above row of products may be written down in any order we please; provided that, in commencing the multiplication by any other figure than the one in the units' place, we use proper caution in placing the product in its proper situation with respect to the decimal point.

Exs. 27. Express as simple decimals

1. 1·5 × 1·5.
2. 2·375 × 3·48.
3. ·006 × 78·928.
4. 1·0006 × 461·8.
5. 10·375 × ·0074.
6. 1·5 × 3·15 × 2·17.
7. ·008 × 5·5 × 1·4.
8. 3·75 × ·014 × ·875.
9. 3·125 × 14·25 × ·01.
10. 35·01 × 7·98 × 1·0001.

118. Next, let the decimals to be multiplied together be either one or both of them circulating. These may be converted into equivalent fractions; and then, after having multiplied them, we may convert the product into a circulating decimal; or the product of the decimals themselves may be found by Simple Multiplication, provided that there be taken a sufficient number of figures to enable us to ascertain the period in the product. We will work two Exs. to illustrate both methods.

Ex. V. Find the value of $\cdot\dot{3}\dot{6} \times 75$.

By the first method I have

$$\cdot\dot{3}\dot{6} \times 75 = \frac{\overset{4}{\cancel{36}}}{\underset{11}{\cancel{99}}} \times 75 = \frac{300}{11} = 27\cdot2727\ldots\ldots = 27\cdot\dot{2}\dot{7}.$$

By the second method, I write down the period three times, so that there may be figures enough in the product to shew its period.

$$\begin{array}{r} \cdot363636\ldots \\ 75 \\ \hline 1818180 \\ 2545452 \\ \hline 27\cdot272700 \\ \hline\hline \end{array}$$

If a fourth period had been written in the multiplicand, the figures carried from the multiplication of it would have caused the period 27 to have been seen in the fifth and sixth decimal places in the product: and every additional period in the multiplicand would have produced one more period in the product: hence, since the number of periods, 36, in the multiplicand is unlimited, so, also, will be the number of periods, 27, in the product; *i. e.* the product is $27\cdot\dot{2}\dot{7}$, as before.

Ex. VI. Find the value of $37\cdot\dot{3} \times 9\cdot1\dot{6}$.

Upon working this Ex. by ordinary Multiplication, I find that even if I write down five periods in the multiplicand, and four in the multiplier, yet the product is such that the learner would hardly detect the period in it; and if the periods had contained several figures, the work would be exceedingly heavy; and since in such Exs. involving two or more recurring decimals, the former method is the better, I give it alone. I then have

$$37\cdot\dot{3} \times 9\cdot1\dot{6} = 37\tfrac{3}{9} \times 9\tfrac{15}{90} \qquad \text{by (110)}$$

$$= 37\tfrac{1}{3} \times 9\tfrac{1}{6}$$

$$=\frac{\overset{56}{\cancel{112}}}{3}\times\frac{55}{\underset{3}{\cancel{6}}}$$

$$=\frac{3080}{9}=342\cdot\dot{2}.$$

Exs. 28. Form the following products:

1. $\cdot\dot{3}\times 1\cdot 78$.
2. $1\cdot\dot{2}\dot{7}\times\cdot 0458$.
3. $13\cdot 0\dot{0}7\dot{9}\times 4\cdot 5$.
4. $27\cdot\dot{3}\times 4\cdot\dot{5}$.
5. $1\cdot\dot{6}\times\cdot\dot{3}48\dot{5}$.
6. $3\cdot 58\times\cdot\dot{2}\dot{1}\times 4\cdot 1\dot{6}$.

DIVISION.

119. In the Exs. under this head, either dividend or divisor, or both, may be a decimal. I will give one Ex. of each variety.

Ex. Find the value of $37\cdot 5\div 84$.

Where, as in this Ex., the divisor is a composite number containing no prime factor greater than 12, we can divide by successive factors, as in (100); thus, dividing first by 12, and then by 7,

$$\frac{37\cdot 5}{84}=\frac{3\cdot 125}{7}=\cdot 44642857142,\ \&c.$$

$$=\cdot 446\dot{4}2857\dot{1}.$$

120. Where there are decimals in the divisor, we have, as in Part I., the following general rule:

Rule. Count the number of decimal places in the divisor; count also the same number in the dividend, and make a mark (ˋ) cutting off all the remaining figures of the dividend. If there be not as many places in the dividend, as in

-r, add ciphers to make up the number, and then

ark.

Divide as in whole numbers; all the figures in the quotient obtained by using the figures in the dividend to the left of the mark will be whole numbers. When these figures have been used, place the decimal point in the quotient; the remaining figures obtained by division will be decimals.

121. Obs. It may happen that the divisor is larger than that part of the dividend to the left of the mark; in this case there will be no whole numbers in the quotient. And if, when the mark is passed, the divisor is still too large, a cipher must be placed in the quotient after the point, for every figure that is used beyond the mark, until the dividend becomes large enough to contain the divisor, and therefore until a figure, other than 0, appears in the quotient.

And this will be seen at once to be correct, if we observe that the part of the dividend so cut off to the left, and the whole divisor, may be considered as thereby reduced to the *same* name, viz. the lowest in the divisor, as tenths, hundredths, as the case may be; hence the division up to the mark will give a whole number; and if continued beyond the mark, the dec^l point must be then placed, and the remaining figures of the quotient will be decimals.

In the following example I make the mark (ʻ) so as to retain *four* decimals to the left of it, because there are *four* in the divisor, and work precisely as with integers, placing the dec^l point in the quotient, as soon as the mark has been passed.

Ex. III. Find the value of 41·0632884 × ·0438.

$$\begin{array}{l}
\cdot0438)\ 41\cdot0632`884\ (937\cdot518 \\
\quad\quad\quad\ \ \underline{3942} \\
\quad\quad\quad\ \ \ \ 1643 \\
\quad\quad\quad\ \ \ \ \underline{1314} \\
\quad\quad\quad\ \ \ \ \ \ 3292 \\
\quad\quad\quad\ \ \ \ \ \ \underline{3066} \\
\quad\quad\quad\ \ \ \ \ \ \ \ 2268 \\
\quad\quad\quad\ \ \ \ \ \ \ \ \underline{2190} \\
\quad\quad\quad\ \ \ \ \ \ \ \ \ \ \ \ 788 \\
\quad\quad\quad\ \ \ \ \ \ \ \ \ \ \ \ \underline{438} \\
\quad\quad\quad\ \ \ \ \ \ \ \ \ \ \ \ 3504 \\
\quad\quad\quad\ \ \ \ \ \ \ \ \ \ \ \ \underline{\underline{3504}}
\end{array}$$

In the next example, since there are *two* places in the divisor, and the dividend is a whole number, I place the dec^l point after the dividend, and append *two* ciphers, and then place the mark.

Ex. IV. 9717÷1·23.

```
1·23) 9717·00' (7900
      861
      1107
      1107
         00
```

Sometimes, also, though there be a sufficient number of decimal places in the dividend to give one or two places in the quotient, yet if it *is* required that there should be 5 or 6 decimal places in the quotient, ciphers may be appended to the dividend, as before, and the division continued as far as we please.

Ex. V. 62·5÷·025.

For the sake of illustration, I work this Ex. in two ways, independently of any rule; first, by removing the point three places to the *right* in the numerator and denominator, and secondly, by converting the given decimals into fractions, and performing the division by the usual method of inverting the divisor.

$$\frac{62·5}{·025}=\frac{62500}{25}=\frac{12500}{5}$$

$$=2500$$

$$\text{or, }\frac{62·5}{·025}=\frac{625}{10}\div\frac{25}{1000}=\frac{\overset{25}{\cancel{625}}}{\cancel{10}}\times\frac{\overset{100}{\cancel{1000}}}{\cancel{25}}$$

$$=2500, \text{ as before.}$$

Exs. 30. Find the required quotients in the following examples:—

1. 13·5÷·15.
2. 83·75÷·128.
3. 1080÷·008.
4. 345·6÷1·728.
5. 13·358697÷·634.
6. ·084007÷34·3.
7. 576·84325÷119·3.
8. 3·84765÷1536.
9. 1·0005÷106·3.

122. If either or both of the given decimals circulate, the circulator may be converted into a proper or improper frⁿ, as the case may be, and the division proceeded with as in fractions; or the circulator may be left unaltered, if in the dividend, and only the divisor be converted into a frⁿ, and the division then performed.

Ex. I. Find the value of $75 \div \cdot\dot{1}4\dot{8}$.

By (108) $\cdot\dot{1}4\dot{8} = \frac{148}{999} = \frac{4}{27}$ (in lowest terms)

$$\text{therefore } \frac{75}{\cdot\dot{1}4\dot{8}} = \frac{73}{\frac{4}{27}} = \frac{75 \times 27}{4} = \frac{2025}{4} = 506\cdot25.$$

Ex. II. Find the quotient of $\cdot96\dot{3}4\dot{5}$ when divided by $\cdot\dot{3}$.

$$\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$$

$$\text{therefore } \frac{\cdot96\dot{3}4\dot{5}}{\cdot\dot{3}} = \frac{\cdot96\dot{3}4\dot{5}}{\frac{1}{3}} = (\cdot96345345\ldots) \times 3$$

$$= 2\cdot89036036\ldots$$

$$= 2\cdot89\dot{0}3\dot{6}.$$

123. It has been stated in (113) and (121) that ciphers may be written after the last figure of a decimal without altering its value: similarly, they may be cut off without affecting it; and this is generally done, if the decimal resulting in any Ex. have ciphers at the end. Thus in Art. 116, Ex. II., I might have removed the cipher which is at the end of the final decimal; only that a pupil would have thought that there were pointed off only 11 decimal places, instead of 12: but, by referring to the last vulgar fraction used in that Ex., I find that it might have been reduced to lower terms, by dividing numerator and denominator by 10; and there would then have been but 11 ciphers in the denʳ, and consequently 11 places in the decimal which follows.

Exs. 32. Find the required quotients in the following Examples:—

1. $\cdot\dot{3}\dot{6} \div \cdot072$.
2. $27\cdot\dot{5} \div \cdot0\dot{6}$.
3. $34\cdot7\dot{5} \div 1\cdot\dot{5}\dot{6}$.
4. $1\cdot23123\ldots \div 3\cdot\dot{6}\dot{3}$.
5. $\cdot18 \times \cdot09 \div \cdot1\dot{6}$.
6. $13\cdot\dot{7}\dot{5} \div (1\cdot\dot{3} + 5\cdot\dot{6})$.

REDUCTION OF DECIMALS.

124. We have here to perform in Decimals the operation which in (62) was performed in Vulgar Fractions; and this merely requires that the decimal quantity should be reduced to successive lower denominations, until we have either no decimal part remaining, or until we reach the lowest denomination used.

Ex. I. Express in positive terms ·375 of £1.

$$\cdot375 \text{ of } \pounds1 = \cdot375 \times 20s. = 7\cdot5\cancel{0}\cancel{0}s.\,;$$
$$\text{and } \cdot5s. = \cdot5 \times 12d. = 6\cdot0d. = 6d.\,;$$
$$\text{therefore } \cdot375 \text{ of } \pounds1 = 7s.\ 6d.$$

$$\begin{array}{r} \pounds \\ \cdot375 \\ \underline{20} \\ 7\cdot5\cancel{0}\cancel{0}s. \\ \underline{12} \\ \underline{\underline{6\cdot0d.}} \end{array}$$

The work may be written out as annexed: where it is plain that at the end of each successive multiplication I point off as many decimals as there were in the preceding line, because there are no decimal places in the multiplier.

It will be seen that in both the above operations I might have omitted two ciphers at the end of the decimal part in the shillings, for the reason given at the close of the last article. I have therefore crossed them out.

By using the equalities mentioned at the end of (101), and referring to the aliquot parts of different denominations given in (64), we might have worked this Ex. very briefly.

$$\text{Thus, } \cdot375 \text{ of } \pounds1 = \tfrac{3}{8} \text{ of } \pounds1 = 7s.\ 6d.$$
$$\text{So, also, } \cdot875 \text{ of } \pounds1 = \tfrac{7}{8} \text{ of } \pounds1 = 17s.\ 6d.$$

Ex. II. Find the value of 7·14685 of 5*s*. 6¾*d*.

Here, since the concrete number is expressed in several denominations, I must either reduce the decimal to a fraction, or reduce the 5*s*. 6¾*d*. to the fraction of a penny, and then perform the multiplication.

By the second method, giving only the principal steps of the work:

$$\begin{aligned} 7\cdot14685 \times 5s.\ 6\tfrac{3}{4}d. &= 7\cdot14685 \times 66\tfrac{3}{4}d. \\ &= 7\cdot14685 \times 66\cdot75d. \\ &= 477\cdot0522375d. \\ &= 39s.\ 9\cdot0522375d. \\ \textit{Answer.} &= \pounds1\ 19s.\ 9\cdot0522375d. \end{aligned}$$

Just as in (63) I left the remainder as a fractional part of a penny, so here I leave it as a decimal fraction of a penny.

125. If any circulating decimals occur in Exs. under this head, we must reduce them to fractions, and proceed as before.

Ex. III. Find the value of $\cdot\dot{5}$ of £1 11*s.* 4*d.*

$$\cdot\dot{5} \text{ of £1 } 11s.\ 4d. = \frac{5}{9} \text{ of } 31\tfrac{1}{3}s.$$
$$= \frac{5}{9} \times \frac{94}{3}s. = \frac{470}{9 \times 3}s. = \frac{156\cdot666...}{9}s.$$
$$= 17\cdot4074074...s.$$

and $\cdot407407...s. = (\cdot407407...) \times 12d.$
$= 4\cdot888884...d.$
$= 4\cdot\dot{8}d.$
$= 4\frac{8}{9}d.$

therefore $\cdot\dot{5}$ of £1 11*s.* 4*d.* = 17*s.* $4\frac{8}{9}$*d.*

Exs. 33. Express in positive terms

1. $1\cdot375$ of 1 guinea.
2. $\cdot028$ of a moidore.
3. $375\cdot794$ of £5.
4. $1\cdot115$ of a crown.
5. $\cdot148325$ of 7*s.* 6*d.*
6. $49\cdot864$ of £15 10*s.*
7. $\cdot05$ of $7\frac{1}{2}$*d.* + $\cdot375$ of 3*s.* 6*d.*
8. $11\cdot05$ of £2 — $17\cdot5$ of 6*s.* 8*d.*
9. $\cdot375$ of a mile + $7\cdot5$ of a yard.
10. $1\cdot185$ of a cwt. — $\cdot0375$ of a qr.
11. $\cdot\dot{5}$ of 6*s.* 8*d.* + $1\cdot\dot{2}\dot{7}$ of a guinea.
12. $\cdot\dot{0}3\dot{7}$ of 27*s.* — $\cdot02$ of £2.

126. The following operation is the converse of that performed in the last two Arts., and corresponds to that exhibited in (78) in Vulgar Fractions.

Ex. Reduce 4*s.* $6\frac{1}{2}$*d.* to the decimal of a sovereign.

In working this Ex. we shall first reduce the former of the given quantities to the fraction of the latter, and then convert into a decimal the vulgar fraction connecting the two quantities.

$$\frac{4s.\ 6\frac{1}{2}d.}{£1} = \frac{54\frac{1}{2}d.}{240d.} = \frac{54\cdot5}{240} = \frac{4\cdot54166...}{20} = \frac{2\cdot27083\dot{3}}{10} = \cdot227083\dot{3},$$

or 4*s.* $6\frac{1}{2}$*d.* = $\cdot227083\dot{3}$ of £1.

The mode most commonly adopted for obtaining this result is the converse of the operation exhibited in the margin of (124); thus,

$$\begin{array}{r|l} 2 & 1\cdot \\ \hline 12 & 6\cdot5 \\ \hline 20 & 4\cdot54166\ldots \\ \hline & \cdot2270833\ldots \end{array}$$

where, in the second line, after the division by 2 to bring the halfpenny into a decimal of 1*d.*, the 6 is placed before the point, giving a dividend 6·5*d.*; and in the third line, after the division by 12, the 4, denoting 4*s.*, is placed before the point.

This method is perhaps the readiest in practice; but I think it is seldom performed by pupils otherwise than mechanically; and it is not suitable to many of the Exs. given below; indeed only to such as merely involve a change from one simple denomination to another in the Tables of money, &c.

Exs. 34.

1.	Reduce	2*s.* 6*d.*	to the decimal fraction of	15*s.*
2.	„	3*s.* 7*d.*	„ „	£5.
3.	„	25*s.*	„ „	3 guineas.
4.	„	£17 15*s.* ..	„ „	£100.
5.	„	354 yds.	„ „	1 league.
6.	„	11440 yds...	„ „	2 acres.
7.	„	3½ qrs.	„ „	15 tons.
8.	„	6 hours	„ „	135 days.
9.	„	1 leap year	„ „	3 weeks 4 days.
10.	„	⅗ of a mark	„ „	$\frac{9}{11}$ of a crown.

MISCELLANEOUS EXAMPLES.

127. The same remark applies to these Miscellaneous Examples in Decimals that applied to the corresponding Exs. in Fractions: and the only general assistance that can be given to a pupil is to shew him the neatest method of performing the operations required.

Ex. I. Multiply £8 17*s.* 6*d.* by 75·25, and reduce the result to the decimal of £100.

$$\text{Here } \frac{(\pounds 8\ 17s.\ 6d.)\times(75\cdot25)}{\pounds 100}=\frac{\pounds 8\tfrac{7}{8}\times 75\cdot25}{\pounds 100}$$

$$=\frac{8\cdot875\times75\cdot25}{100}$$

$$=\frac{667\cdot84375}{100}$$

$$=6\cdot6784375.$$

Ex. II. Reduce $\frac{3\cdot 275}{405}$ of $\frac{2\cdot 5}{\cdot 075}\times\frac{3\cdot 125}{11}\times\frac{9}{9\cdot 375}$ to a simple quantity.

Moving the point three places to the right in two numerators and in two denominators, and again one place to the right in the numerator and the denominator of the second fraction, the expression becomes

$$\frac{\overset{\overset{131}{\cancel{655}}}{\cancel{3275}}}{\underset{\underset{9}{\cancel{81}}}{\cancel{405}}}\times\frac{\cancel{25}}{\underset{\underset{6}{\cancel{30}}}{\cancel{750}}}\times\frac{\cancel{3125}}{11}\times\frac{\cancel{9}}{\underset{3}{\cancel{9375}}}=\frac{131}{9\times 6\times 11\times 3}$$

$$=\frac{11\cdot 90909\ldots}{9\times 6\times 3}$$

$$=\frac{1\cdot 32323232\ldots}{6\times 3}$$

$$=\frac{\cdot 22053872053872\ldots}{3}$$

$$=\cdot 07351290684624\ldots,\ \&c.$$

Since a very large number of decimals is non-terminating, it might seem that vulgar fractions, which are always expressed in finite terms, would be preferable for every purpose. But this is not the case, for decimals have one advantage over fractions from the following consideration.

In ascertaining the comparative value of two or more fractional quantities, if they be expressed as vulgar fractions, it is necessary to reduce them to a common denominator; but if they are represented as decimals, mere inspection will detect their comparative value, as readily as can be done in whole numbers.

For example, if we have to compare $\frac{7}{15}$, $\frac{11}{25}$, $\frac{13}{28}$, we cannot see which is the greatest, and which the least, without reducing them to a common denominator: but if the quantities had been written in their decimal form, viz.

$$\cdot 4666\ldots\ldots\ldots;\qquad \cdot 44;\qquad \cdot 46\dot{4}2857\dot{1},$$

we could see *at once* that the first is the largest; the last one is the next; and the middle one is the smallest: therefore, in order of magnitude they are $\frac{7}{15}$, $\frac{13}{28}$, $\frac{11}{25}$. The decimal form is more especially useful, when several fractional quantities are arranged in a table, and where it is requisite to be able to compare the different quantities at a glance.

We may take as an Ex. the accompanying table, in which the numbers represent the comparative weights of equal bulks of different substances; or, as they are generally termed, their Specific Gravities.

Sheet Glass	3·33
Plate Glass	2·5
Marble	2·716
Quartz	2·6
Rock Salt	1·92
Ivory	1·917
Ice at 0°	·926
Water at 60°	1·

Though the 5th and 6th numbers are very nearly equal, yet it can be seen at once that the 5th is larger than the 6th, by three thousandths: as fractions, these quantities would have been written, $1\frac{23}{25}$; and $1\frac{917}{1000}$; and the difference could not have been ascertained by inspection.

Exs. 35.

1. Express as a simple decimal the difference between $\frac{3}{8}$ of $\frac{11}{12}$ and $\frac{7}{8}$ of $\frac{9}{14}$.

2. Simplify the following expression:

$$\frac{1{\cdot}5}{{\cdot}075} \times \frac{3{\cdot}25}{1\frac{1}{2}} + \frac{1{\cdot}875}{2{\cdot}1} \times \frac{3{\cdot}5}{3{\cdot}75}.$$

3. How many times must ·35 of a groat be repeated, to produce $\frac{1}{2\frac{6}{7}}$ of a crown?

4. Form the following quantities into a decimal table, as in (127), arranging them in order of magnitude:

$$2\frac{7}{8}, \qquad 1\frac{5}{8} \text{ of } 1{\cdot}06, \qquad \frac{2}{7{\cdot}5}, \qquad 1{\cdot}75 \text{ of } 3.$$

5. What is the average value of the above quantities, expressed as a vulgar fraction?

6. Find the simple decimal equivalent to $1{\cdot}\dot{3}\dot{1} \times (2{\cdot}\dot{4} + 7{\cdot}\dot{5})$.

7. Express in positive terms the sum of 1·05 of a crown, $\cdot\dot{7}1428\dot{5}$ of a guinea, and $\cdot1\dot{6}$ of 6*s.* 8*d.*

8. Convert $\frac{5}{16}$ into a decimal, by multiplying both num^r and den^r by some common quantity.

9. Reduce $2\frac{3}{8}$ of $14\frac{1}{2}$ acres to the decimal of a square mile.

10. Find in a decimal form a fourth proportional to each of the following sets of numbers:

1, ·2, ·375; 3·25, ·0175, 1·01; $\frac{11\frac{1}{2}}{3}$, $\frac{3}{7}$, 2·75.

Exs. 36. **E.**

1. The product is 154923000, and multiplicand 42375; what is the multiplier?

2. How many calendar months in $3\frac{1}{4}$ centuries?

3. If 90 degrees=100 grades, find the number of degrees in $12\frac{1}{2}$ grades.

4. A man enters into business with £20,000, and each year makes a profit of one-fourth of his investment, and adds that profit to his capital; how much will he be worth in 5 yrs.?

5. How many fathoms are there in a degree?

6. If there are 360° in every circle, and in latitude 30° a degree =34·75 miles, what is the length of the circle which passes through latitude 30°?

7. What is the average length of the calendar months, including leap year?

8. Find the abstract number which expresses the ratio of £$12\frac{7}{8}$ and $17\frac{3}{8}$ shillings.

9. Assuming that the number of square feet in the area of an oblong surface is found by multiplying the number of feet in the length by the number in the breadth, find how many bricks, each 9 in. by $4\frac{1}{2}$ in., are required to pave a floor $97\frac{1}{2}$ ft. long, and 81 ft. broad.

10. Write an equation involving the signs of add^n, sub^n, mult^n, and div^n, and having the right-hand side=237.

11. Explain the object and the process of reducing fractions to a C. D. Reduce to L. C. D. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, explaining the work.

12. Convert into a vulgar fraction, in lowest terms, the sum of 9 tenths, 3 hundredths, and 5 thousandths.

13. Multiply the difference between 75 hundredths and 5 tenths, by 8 thousandths; expressing the result as a vulgar fraction.

14. Reduce the following expression to a simple decimal:—

$$\frac{1}{2}(6\frac{1}{2}+2\frac{3}{8}-3).$$

15. What decimal of £1 is $\frac{3}{8}$ of 13*s.* 4*d.*?

F.

1. There are 100 links in a chain of 4 perches; how many links in a league?

2. The wheel of a locomotive which is $16\frac{1}{2}$ ft. in circumference, turns round 25764 times, how many miles will it have run?

3. Find the cost of half an acre of building land, at 3*s*. 6*d*. per sq. yd.

4. What is the expense of painting the walls of a room, 17 feet long, 13 ft. broad, and 10 ft. high, at 7*d*. a square yard?

5. On a railway, from *A* to *B* there is a rise of 1 in 160 for $\frac{3}{4}$ mile, of 1 in 330 for 2 miles 50 yards, then a fall of 1 in 110 for 1 mile 360 yards, lastly, a rise of $\frac{1}{320}$ for 1500 yards; what is the height of *A* compared with *B*?

6. A general sends away $\frac{13}{25}$ of his army, and then $\frac{3}{8}$ of the remainder; he has now 1350 men, what had he at first?

7. Find the exact value of $\cdot 1\dot{6}$ of a moidore + $\cdot\dot{5}7142\dot{8}$ of 2 guineas.

8. How many allotments, each 2r. 25p., are contained in $47\frac{1}{4}$ acres?

9. If there are 100 links in a chain of 22 yards, find how many square links are contained in an acre.

10. A bankrupt owes £3840, and his property amounts to only £1656; how much will his creditors receive in the pound?

11. If a board be $23\frac{1}{2}$ inches broad, how long must it be to contain 15 square feet?

12. If 500 slates would cover a surface 25 feet square, how many would be required for a roof 27 ft. by 36?

13. What number divided by $17\frac{1}{5}$ will give $13\frac{7}{8}$?

14. To $\frac{9}{16}$ of a gross add $\frac{5}{8}$ of a quarter of a hundred; from this sum subtract $\frac{7}{8}$ of a score, and divide the remainder by $17\frac{1}{5}$.

15. Simplify the expression $\frac{3}{8}$ of $\frac{10}{27}$, proving that the mode of working is correct; and explain what is called *cancelling*.

G.

1. If the interest of the national debt be £1 2*s*. 6*d*. each second of time, what is the amount per annum?

2. How many bottles in 103 casks, each containing $9\frac{3}{4}$ dozen?

3. A person pays a debt of £230 8*s*. in sovereigns, half sovereigns, crowns and shillings, of each an equal number; how many of each?

4. What is the amount of the following items, $54\frac{1}{4}$ lbs. at 5*s*. $4\frac{1}{2}$*d*., 241 yds. at $7\frac{1}{2}$*d*., and 512 pieces at 2*s*. $4\frac{3}{4}$*d*.?

5. Express as a Vulgar Fraction, in its lowest terms, the sum of 7 hundredths, 5 thousandths, and 375 tenths of thousandths.

6. How much paper $1\frac{1}{4}$ yard broad, will cover as much as 20 yards, of $\frac{3}{4}$ yd. broad?

7. If 4 lb. of silver be mixed with 4 lb. 5 oz. of gold, how much silver will there be to 6 oz. of gold?

8. If one whose rent is £430 pays a tax of £30 6*s.* 9*d.*, what should be the rent of a man whose taxes come to £94 16*s.* 1$\frac{1}{8}$*d.*?

9. Gunpowder being composed of nitre 15 parts, charcoal 3 parts, and sulphur 2 parts; find how much of each is required in making 16 cwt. of powder.

10. I take successively $\frac{1}{3}$ and then $\frac{1}{4}$ of a sum of money, and find that I have left £15; what was the sum at first?

11. Explain the two methods of multiplying a fraction by a whole number, taking as an Ex. $\frac{5}{8}\times 3$.

12. In a bridge of 7 arches, the middle one is 75 feet span, and the others on each side are $\frac{1}{10}$th less in each succeeding arch; find the whole length of the bridge, allowing 15 feet for each pier.

13. Find the value of $\left(\frac{4}{5\frac{1}{2}}\text{ of £5}\right) \sim 3\cdot\dot{4}65\dot{3}$ of a guinea.

14. What is the price per lb. of an article, of which 1·5 cwts. cost 113·75 shillings?

15. Express in positive terms $\cdot 37\dot{6}$ of 27*s.* 6*d.* $\sim 1\cdot\dot{8}$ of 2 guineas.

H.

1. How many bottles of wine in 12 pipes, at the rate of 52 dozen 9 bottles each pipe?

2. Find the number of square yards in an area, $\frac{1}{2}$ a mile long, and $\frac{1}{4}$ of a mile broad.

3. If one man thrash 7 sheaves of corn in a day, and each sheaf yield 3$\frac{1}{2}$ pecks, and each peck 15 lbs.; how much in quantity and weight will 15 men thrash in 6 weeks?

4. A general having an army of 24,000 men, increases it one-third by recruiting; afterwards he loses one-fourth by disease, and of the remainder one-fifth fell in battle; how many men has he left?

5. How many fathoms in a degree?

6. What is the ratio of a geographical mile to a British mile? How many geographical miles must I measure, so as to contain the least exact number of British miles?

7. If a person step at an average $2\cdot 1\dot{6}$ feet, how many steps must he take in ·325 miles?

8. Reduce $\frac{7}{256}$ to a decimal without using Long Division; and shew that $\frac{3}{125}=\cdot 024$, without dividing at all.

9. Find the value of $4\cdot 25 \div \cdot 10625$, proving the result by Vulgar Fractions.

10. Write in words ·75, 2·0324, 17·000001.

11. Find the value of (£25 16*s.* 7$\frac{3}{5}$*d.*) × 8÷$\frac{11}{12}$.

12. The circumferences of the fore and hind wheels of a carriage are respectively $9\frac{3}{7}$ ft. and $13\frac{3}{11}$ ft.; find how many more revolutions one makes than the other in $10\frac{3}{8}$ miles.

13. What fraction of a guinea and a half, together with £3 16*s.* 9*d.*, will give £4?

14. Exhibit as a simple vulgar fraction the result of

$$\cdot 3\dot{7} \div 1\cdot 17 \times \cdot 052 \div 1\cdot\dot{8}.$$

15. Find the ratio between the product and quotient of 147 and $\cdot\dot{2}\dot{7}$.

PRACTICE.

128. **Practice** is a rule which endeavours to shew the readiest method of finding the cost of any number of articles at a certain price: and the work exhibited in any Ex. consists of a series of amounts such as a person would try to obtain, if he were working the question mentally.

Thus, if I had to find the value of 54lbs. at $1\frac{1}{2}$*d.* each, I should say, 54 at 1*d.* = 54*d.* = 4*s.* 6*d.*; and 54 at $\frac{1}{2}$*d.* = 54 halfpence = 27*d.* = 2*s.* 3*d.*: and therefore 54 at $1\frac{1}{2}$*d.* = 4*s.* 6*d.* + 2*s.* 3*d.* = 6*s.* 9*d.*

129. But if the number of articles had been much larger, or the price much greater, the value of them could not readily have been obtained mentally: we therefore, in Practice, use the above method of mental calculation, but we write down the successive results, and find their sum for the final result.

The Exs. to be worked under this Rule may be arranged as follows:—

When the price is under a shilling, as

Ex. I. 4108 at $7\frac{3}{4}$*d.*

When the price is between 1*s.* and £1, as

Ex. II. 4103 at 7*s.* $5\frac{1}{4}$*d.*

Ex. III. 6009 at 19*s.* $5\frac{1}{4}$*d.*

When the price consists of more than £1, as

Ex. IV. 7111 at £1 17*s.* 4$\frac{1}{2}$*d.*

Ex. V. 4013 at £12 7*s.* 0$\frac{1}{2}$*d.*

When there is a fraction in the given number of quantities, as

Ex. VI. 6583$\frac{8}{15}$ at £1 19*s.* 11$\frac{3}{4}$*d.*

When there is a fractional part of a penny other than farthings, as

Ex. VII. 4176 at £3 5*s.* 4$\frac{9}{10}$*d.*

When the quantity, the price of which is required, consists of several denominations, as

Ex. VIII. 9lbs. 3oz. 14dwt. at £10 15*s.* 6*d.* per lb.

130. Mention was made in (64) of certain fractional parts of £1, 1*s.*, &c., which were termed aliquot parts of £1, 1*s.*, &c. It is advisable to have such parts of the denominations most in use familiarly in the mind; but a pupil will find in PRACTICE, that he has to take aliquot parts of many other quantities which are intermediate between such standard units as £1, 1cwt., &c.: and nothing will render him expert in taking such aliquot parts as he will require, but a readiness in the treatment of fractions.

The following are the most useful aliquot parts of £1 and 1*s.*, and should therefore be remembered.

10*s.* 0*d.* = $\frac{1}{2}$ £
6*s.* 8*d.* = $\frac{1}{3}$,,
5*s.* 0*d.* = $\frac{1}{4}$,,
4*s.* 0*d.* = $\frac{1}{5}$,,
3*s.* 4*d.* = $\frac{1}{6}$,,
2*s.* 6*d.* = $\frac{1}{8}$,,
1*s.* 8*d.* = $\frac{1}{12}$,,
1*s.* 4*d.* = $\frac{1}{15}$,,
1*s.* 3*d.* = $\frac{1}{16}$,,
1*s.* 0*d.* = $\frac{1}{20}$,,

6*d.* = $\frac{1}{2}$ shilling.
4*d.* = $\frac{1}{3}$,,
3*d.* = $\frac{1}{4}$,,
2*d.* = $\frac{1}{6}$,,
1$\frac{1}{2}$*d.* = $\frac{1}{8}$,,
1*d.* = $\frac{1}{12}$,,

In all cases we shall find, that where the price given is not an aliquot part of the unit of the next higher denomination, it is necessary to split the price into two or more portions, of which the largest must be an aliquot part of this said unit, and the remaining portions are aliquot parts either of this same unit, or of some one of the portions already used in the Ex.

Ex. I. 4108 at 7¾*d.*

In this Ex. since the price 7¾*d.* is not an aliquot part of 1*s.*, I therefore break up this price into three parts, 6*d.*, 1½*d.*, and ¼*d.*, of which the largest, 6*d.* = ½ of 1*s.* the next higher denomination; the next, 1½*d.* is an aliquot part of 6*d.*, viz. ¼th; and the last, ¼*d.* is ⅙th of 1½*d.* Now, I know that 4108 articles at 1*s.* each would cost 4108*s.*; therefore, 4108 at 6*d.*, *i. e.* at ½*s.*, cost 4108 times ½*s.*, or ½ of 4108*s.*: I therefore find the value of this quantity and write it down, viz. 2054*s.* So also, 4108 at 1½*d.* = 4108 at ¼ of 6*d.*, and therefore = ¼th of the value of 4108 at 6*d.*, which was found just before Similarly, since ¼*d.* = ⅙th of 1½*d.*, therefore 4108 at ¼*d.* = ⅙th of the previously found value of 4108 at 1½*d.*

Ex. I. 4108 at 7¾*d.*

			4108		
A price 6*d.*	= ½*s.*	gives	2054		
1½*d.*	= ¼ of 6*d.*	,,	513		6
¼*d.*	= ⅙ of 1½*d.*	,,	85		7
7¾*d.*		2,0	265,3		1
			£132	13	1

Hence the sum of my three amounts at 6*d.*, 1½*d.*, and ¼*d.*, will be the total value, at 7¾*d.* The whole of the operations which have just been described are to be written out in the accompanying shape.

Had the price been below 6*d.*, as for instance 3175 at 2¾*d.*, we should have taken as parts 2*d.* = ⅙*s.*; ½*d.* = ¼ of 2*d.*; and ¼*d.* = ½ of ½*d.*; and the remainder of the work as before.

In the second division in Ex. I. 4 being a divisor, I had to find the value of $\frac{2054s.}{4} = 513\frac{1}{2}s. = 513s.\ 6d.$; and generally, when, as in (64), a number of shillings, or pounds, is divided by a divisor, and a remʳ is left, the fractional quotient can be converted into positive terms *at once*, and more readily than by the usual method, in which the remainder is reduced to lower denⁿ, and the division again performed. But in the next line, where the divisor is 6, we have to take a sixth part of the 6*d.* as well as of the 513*s.*, hence I have $\frac{513}{6}s. = 85\frac{3}{6}s. = 85s.\ 6d.$; and this, with the sixth part of 6*d.*, viz. 1*d.*, becomes 85*s.* 7*d.* Though it takes a long time to explain these processes, yet a pupil who is quick at working fractions will obtain the above results more readily than I can describe them, and much time will be saved by their use: but those who prefer the usual method of reducing the remainders, as in Compound Division, can of course adhere to it. The 2653 in the result evidently consists of shillings; and from this Ex. we see that the highest denomination in the sum of all the separate amounts is the same as that of which we took the first aliquot parts; and so also is the first remainder obtained in each of the divisions.

Exs. 37.

		d.			*d.*
1.	1857 at	¼.	7.	6329 at	10.
2.	7151 ,,	¾.	8.	8537 ,,	8¾.
3.	3982 ,,	1¾.	9.	11071 ,,	10½.
4.	6110 ,,	5¼.	10.	7705 ,,	10¾.
5.	3457 ,,	9¼.	11.	3956 ,,	11½.
6.	7403 ,,	9½.	12.	8793 ,,	11¾.

Ex. II. 4103 at 7*s.* 5¼*d.*

		4103		
A price 5*s.* 0*d.* = ¼£	gives	1025	15	
2*s.* 0*d.* = $\frac{1}{10}$£	,,	410	6	
4*d.* = $\frac{1}{6}$ of 2*s.*	,,	68	7	8
1*d.* = ¼ of 4*d.*	,,	17	1	11
¼*d.* = ¼ of 1*d.*	,,	4	5	5¾
7*s.* 5¼*d.*		£1525	16	0¾

In breaking the price 7*s.* 5¼*d.* into parts, we find that 2*s.* is $\frac{1}{10}$th of £1, and not of the previous aliquot part 5*s.*; hence, in dividing by 10, we must take as dividend, not the line corresponding to 5*s.* but the top line, 4103, which is the value of 4103 at £1 each: and in taking aliquot parts, we must always be careful to take as dividend, that line which expresses the value given by that coin or denomination of which we are taking an aliquot part.

OBS. In this and the preceding Ex. I have written the denomination of which the aliquot part has been taken in every line: but for the future I shall generally omit the denomination when I am taking an aliquot part of the line preceding, but insert it when I am taking a part of some earlier dividend.

EXS. 38.

		s.	*d.*
1.	7992 at	1	7½.
2.	8495 „	2	9¾.
3.	3708 „	3	7¼.
4.	8222 „	5	10¾.
5.	1007 „	7	11¼.
6.	1173 „	3	8¼.
7.	4351 „	8	0½.
8.	3672 „	7	10.
9.	10205 at	4	10½.
10.	7248 „	9	1¼.
11.	12826 „	8	11¼.
12.	8798 „	9	7¾.
13.	1250 „	9	11½.
14.	7108 „	8	6¾.
15.	11489 „	9	10.
16.	12146 „	9	8¾.

EX. III. 6009 at 19*s.* 5¼*d.*

				£	*s.*	*d.*
				6009		
A price	10*s.* 0*d.*	= ½£	gives	3004	10	
	5*s.* 0*d.*	= ½	„	1502	5	
	4*s.* 0*d.*	= ⅕£	„	1201	16	
	4*d.*	= $\frac{1}{12}$	„	100	3	
	1*d.*	= ¼	„	25	0	9
	¼*d.*	= ¼	„	6	5	2¼
	19*s.* 5¼*d.*			£5839	19	11¼

This Ex. differs from the preceding, only in having the price above 10*s.* In such a case we always take 10*s.* as the first aliquot part; and the remaining shillings and pence as aliquot parts, either of 10*s.*, or of any amount which has been used in the course of the example. Sometimes it may happen that a second aliquot part of £1 is taken, as in 16*s.* 8*d.*, where I should take 10*s.* = ½£, and 6*s.* 8*d.* = ⅓£.

EXS. 39.

		s.	*d.*
1.	18147 at	10	6½.
2.	3501 „	18	9.
3.	6234 „	11	4½.
4.	7646 „	16	0¾.
5.	5431 „	15	11¼.
6.	11040 „	16	10.
7.	2067 „	17	8¾.
8.	3459 „	18	6.
9.	1948 at	19	0.
10.	8218 „	15	10½.
11.	12327 „	17	10¾.
12.	12497 „	19	10¼.
13.	14889 „	18	0¼.
14.	4103 „	14	10½.
15.	12018 „	19	5¼.
16.	8972 „	19	9¼.

Ex. IV. 7111 at £1 17*s.* 4½*d.*

A price £1 0*s.* 0*d.*		gives	7111		
10*s.* 0*d.*	= ½ £	,,	3555	10	
5*s.* 0*d.*	= ½	,,	1777	15	
2*s.* 0*d.*	= $\frac{1}{10}$£	,,	711	2	
4*d.*	= $\frac{1}{6}$	,,	118	10	4
½*d.*	= ⅛	,,	14	16	3½
£1 17*s.* 4½*d.*			£13288	13	7½

Here £7111 is the value of 7111 things at £1 each; therefore, if I find the value of 7111 at 17*s.* 4½*d.*, as in the last Ex., and add in the top line as £7111, I shall obtain a correct result.

Exs. 40.

			£	s.	d.				£	s.	d.
1.	1032	at	1	11	5½.	7.	11000	at	1	4	9¾.
2.	8649	,,	1	0	6¼.	8.	1572	,,	1	12	11½.
3.	15432	,,	1	10	1.	9.	7538	,,	1	11	8¼.
4.	6666	,,	1	8	0½.	10.	19345	,,	1	13	5.
5.	5741	,,	1	17	6.	11.	1548	,,	1	19	7¼.
6.	7891	,,	1	14	11.	12.	8754	,,	1	19	10¼.

Ex. V. 4013 at £12 7*s.* 0½*d.*

			4013		
			12		
A price £12 0*s.* 0*d.*		gives	48156		
5*s.* 0*d.*	= ¼ £	,,	1003	5	
2*s.* 0*d.*	= $\frac{1}{10}$£	,,	401	6	
½*d.*	= $\frac{1}{48}$	,,	8	7	2½
£12 7*s.* 0½*d.*			£49568	18	2½

Here, since £4013 repeated twelve times gives the value of 4013 at £12; therefore I must multiply the 4013 by 12, and add the product as pounds to the other amounts obtained by proceeding with the 7*s.* 0½*d.*, as in Exs. III. and IV. The last division, by 48, cannot of course be performed mentally: a pupil may obtain the result by Long Division, and merely write down the amount.

Exs. 41.

			£	s.	d.				£	s.	d.
1.	6241	at	3	8	7¼.	10.	5482	at	17	15	10¾.
2.	999	,,	4	19	9¼.	11.	7701	,,	6	3	7¾.
3.	5683	,,	5	17	10.	12.	7032	,,	11	3	3¼.
4.	1429	,,	8	19	10¼.	13.	7702	,,	10	17	6½.
5.	4108	,,	17	17	11½.	14.	3764	,,	18	14	7½.
6.	10101	,,	9	9	10.	15.	5505	,,	9	19	11½.
7.	864	,,	20	17	6.	16.	2807	,,	29	0	7½.
8.	1875	,,	13	13	7.	17.	11078	,,	35	15	9.
9.	4273	,,	18	17	8½.	18.	8970	,,	63	14	9.

Ex. VI. $6583\frac{8}{15}$ at £1 19*s*. $11\frac{3}{4}$*d*.

A price £1 0*s*. 0*d*. gives	6583			
10*s*. 0*d*. $=\frac{1}{2}$£ ,,	3291	10		
5*s*. 0*d*. $=\frac{1}{2}$,,	1645	15		
4*s*. 0*d*. $=\frac{1}{5}$£ ,,	1316	12		
10*d*. $=\frac{1}{6}$ of 5*s*.	274	5	10	
1*d*. $=\frac{1}{10}$,,	27	8	7	
$\frac{1}{2}$*d*. $=\frac{1}{2}$,,	13	14	$3\frac{1}{2}$	
$\frac{1}{4}$*d*. $=\frac{1}{2}$,,	6	17	$1\frac{3}{4}$	
$\frac{8}{15}$ of £1 19*s*. $11\frac{3}{4}$*d*. $=$	1	1	$3\frac{13}{15}$	
A price £1 19*s*. $11\frac{3}{4}$*d*. gives	£13160	4	$2\frac{7}{60}$	

This Ex. differs from Exs. IV. and V. only in the presence of the fraction $\frac{8}{15}$. I therefore, after having proceeded with the sum, as though the $\frac{8}{15}$ were not there, find the value of $\frac{8}{15}$ at (£1 19*s*. $11\frac{3}{4}$*d*.) *i.e.* of $\frac{8}{15}$ of (£1 19*s*. $11\frac{3}{4}$*d*.) which, according to the method of Ex. IV. in (84) $=$ £1 1*s*. $3\frac{13}{15}$*d*.

The value of the $\frac{8}{15}$ might also have been thus obtained:

$$\frac{8}{15}=\frac{5}{15}+\frac{3}{15}=\frac{1}{3}+\frac{1}{5};$$

I might therefore have taken one-third and one-fifth of £1 19*s*. $11\frac{3}{4}$*d*. and their sum would have amounted to £1 1*s*. $3\frac{13}{15}$*d*., as before.

Exs. 42.

		£	*s*.	*d*.
1.	$14287\frac{1}{5}$ at	10	13	$10\frac{3}{4}$.
2.	$3779\frac{15}{16}$,,	14	4	$4\frac{3}{4}$.
3.	$8976\frac{3}{7}$,,	7	15	$11\frac{1}{2}$.
4.	$4149\frac{2}{11}$,,	0	8	7.
5.	$4538\frac{4}{9}$ at	0	16	6.
6.	$1008\frac{7}{12}$,,	4	8	6.
7.	$2711\frac{9}{16}$,,	7	3	3.
8.	$3714\frac{5}{14}$,,	2	9	$11\frac{1}{4}$.

Ex. VII. 4176 at £3 5*s*. $4\frac{9}{10}$*d*.

	4176		
	3		
A price £3 0*s*. 0*d*. gives	12528		
5*s*. 0*d*. $=\frac{1}{4}$£ ,,	1044		
4*d*. $=\frac{1}{15}$,,	69	12	
$\frac{1}{2}$*d*. $=\frac{1}{8}$,,	8	14	
$\frac{1}{5}$*d*. $=\frac{1}{20}$ of 4*d*.	3	9	$7\frac{1}{5}$
$\frac{1}{5}$*d*. $=\frac{1}{20}$ of 4*d*.	3	9	$7\frac{1}{5}$
£3 5*s*. $4\frac{9}{10}$	£13657	5	$2\frac{2}{5}$

The presence of the fraction $\frac{9}{10}$*d*. is the only point in which this Ex. differs from III. and IV.; and just as we break up $\frac{3}{4}$*d*. into $\frac{1}{2}$*d*. and $\frac{1}{4}$*d*., so this $\frac{9}{10}$*d*. must be broken up into such portions as will be aliquot parts of 1*d*. viz. $(\frac{5}{10}+\frac{2}{10}+\frac{2}{10})$*d*., or $\frac{1}{2}$*d*. $+\frac{1}{5}$*d*. $+\frac{1}{5}$*d*.

If in this Ex. there had been a fraction at the end of the 4176, as in Ex. VI., we should have proceeded with it just as with the $\frac{8}{15}$ in that Ex.

Exs. 43.

		£	*s*.	*d*.
1.	5189 at	1	10	$2\frac{1}{7}$.
2.	7485 ,,	4	5	$9\frac{3}{5}$.
3.	1111 ,,	14	6	$5\frac{3}{5}$.
4.	$2486\frac{1}{2}$ at		18	$7\frac{1}{5}$.
5.	4321 ,,	1	0	$6\frac{1}{5}$.
6.	4231 ,,		11	$8\frac{1}{5}$.

Ex. VIII. 9 lb. 3 oz. 14 dwts. at £10 15*s.* 6*d.* per lb.

Hitherto we have had the quantities whose value was required expressed all in one denⁿ; and we could therefore repeat the highest denⁿ of the price, as for instance, £1, as many times as there are units in the given quantity, and then take parts of this highest denⁿ for the remainder of the price. But in Ex. VIII. we cannot place 9 lb. 3 oz. 14 dwt. in the top line, and multiply it by the 10, because the result would not be £10 repeated an exact number of times: I therefore place the £10 15*s.* 6*d.* in the top line, and multiplying it by 9, I obtain the value of 9 lb. at £10 15*s.* 6*d.* per lb.; and the value of the 3 oz. 14 dwt. will be found, by taking the same parts of £10 15*s.* 6*d.* that 3 oz. 14 dwt. are of 1 lb. Thus working, we have

					£		
					£10	15	6
							9
Value of 9 lb.	0 oz.	0 dwt.		is	96	19	6
	3	0	= ¼ of 1 lb.	,,	2	13	10½
		10	= ⅙ of 3 oz.	,,		8	11¾
		2	= ⅕ of 10 dwt.	,,		1	9$\frac{11}{20}$
		2	= ⅕ of 10 dwt.	,,		1	9$\frac{11}{20}$
9 lb.	3 oz.	14 dwt.	=		£100	5	11$\frac{7}{20}$

$$\left(\frac{11}{20}+\frac{11}{20}+\frac{3}{4}+\frac{1}{2}\right)d.=\frac{11+11+15+10}{20}d.$$

$$=\frac{47}{20}d.=2\tfrac{7}{20}d.$$

Also, since 3 oz. 14 dwt. $=3\frac{14}{20}$ oz. $=3\frac{7}{10}$ oz. $=\dfrac{3\frac{7}{10}}{12}$ lb. $=\dfrac{37}{120}$ lb., therefore the Ex. may be written $9\frac{37}{120}$ lbs. at £10 15*s.* 6*d.* per lb.; and it is then similar to Ex. VI.

Exs. 44.

			£	s.	d.	
1.	5 lbs. 10 oz. (Avoirdupois)	.....at	0	6	6	per lb.
2.	93 lbs. 3 oz. 1 dwt. 6 grs.	........,,	4	10	6	,,
3.	800 cwt. 0 qr. 16 lbs.	..............,,	14	19	6¼	per cwt.
4.	99 tuns 3 hhds. 16 gals.	..........,,	128	18	9	per tun.
5.	155 gross and 75	....................,,	1	10	6	per gross.

		£	s.	d.	
6.	135 quarters 7 bushels..........at	2	17	8	per quarter.
7.	6 tons 7 cwt. 2 qrs. 17 lbs. ......„	8	10	7	per ton.
8.	8 years 3 months 20 days ......„	5	7	6⅜	per year.
9.	15 reams 9 quires 6 sheets......„	1	6	9	per ream.
10.	46 days 11 hours 35 minutes ...„	1	1	5½	per day of 12 hours.

See Appendix, Decimal Coinage.

Exs. 45.

MISCELLANEOUS EXAMPLES.

1. What cost 1351 lbs. at 2*s.* 2¼*d.* per lb.?
2. A bankrupt pays 13*s.* 9*d.* in the pound upon £1575, how much money did he divide?
3. Find the value of 2078⅓ yards at 3*s.* 7⅜*d.* per yard.
4. What is the tax on £12345 15*s.* at 3*s.* 7½*d.* in the pound?
5. Find the worth of 24150 rupees at 1*s.* 11¾*d.* each.
6. A gold snuff-box weighed 7 oz. 15 dwts. 15 grs., find its value at £4 5*s.* 6*d.* per oz.
7. If 2*s.* 3*d.* in the pound is paid on an income of £1050; what is the net annual income?
8. My daily expenses are 10*s.* 11¾*d.*; how much can I save out of an income of £250?
9. Find the cost of a silver epergne weighing 175 oz. 14 dwts., at 45*s.* 9*d.* per oz.
10. What is the value of 3r. 17p. 25¼ yds., at £125 per acre?

131. In many Exs. similar in principle to those given above, a knowledge of fractions will enable a pupil to employ very brief methods of working; as, for instance, if I had 317 at 16*s.* 8*d.*, I should say

$$317 \text{ at } 16s.\ 8d. = 317 \text{ at } \frac{5}{6}£ = \frac{1585}{6}£ = 264\tfrac{1}{6}£ = £264\ 3s.\ 4d.$$

Again, to find the value of 754 at 7*s.* 7*d.*

$$754 \text{ at } 7s.\ 6d. = 754 \text{ at } \frac{3}{8}£ = \frac{2262}{8}£ = £282\tfrac{3}{4};$$

$$= £282\ 15s.$$

and 754 at 1*d.* = 754*d.* = 62*s.* 10*d.*

therefore, 754 at 7*s.* 7*d.* = £282 15*s.* + £3 2*s.* 10*d.*

$$= £285\ 17s.\ 10d.$$

This method is especially worth notice in short Exs.

Thus, 97 at $7\frac{1}{2}d. = 97 \times \frac{5}{8}s. = \frac{485}{8}s. = 60\frac{5}{8}s. = £3\ 0s.\ 7\frac{1}{2}d.$

Again, when the price is an even number of shillings, under 20, as 327 at 16*s.*, we may work as follows:

$$327 \text{ at } 16s. = 327 \times \frac{8}{10}£ = \frac{2616}{10}£ = 261\frac{6}{10}£ = £261\ 12s.$$

and this mode of working is comprised in the following rule:

Multiply the given number by half the given price, doubling the first figure to the right-hand for shillings, and calling the rest pounds.

$$\begin{array}{lr} \text{Thus,} & 327 \\ & 8 \\ \hline & £261\ \ 12s. \\ \hline\hline \end{array}$$

We may also observe, that since the cost of 12 things at 1*d.* each = 1*s.*, therefore, that of 12 things at $3\frac{1}{2}d. = 3\frac{1}{2}s. = 3s.\ 6d.$, and of 12 things at $7\frac{3}{4}d. = 7\frac{3}{4}s. = 7s.\ 9d.$; *i. e.* if I have the price of one article in pence and a fractional part of a penny, the price of 12 articles will be expressed by the same figures as shillings and parts of a shilling.

Hence, also, the value of any multiple of 12 things may be readily expressed as above, if the price of one be given in terms of pence.

Ex. Find the value of 96 lbs. at $10\frac{1}{4}d.$

Cost of 12 lbs. at $10\frac{1}{4}d. = 10\frac{1}{4}s. = 10s.\ 3d.$

therefore, cost of 8×12 lbs. at $10\frac{1}{4}d. = 8 \times (10s.\ 3d.) = 82s.$

$= £4\ 2s.$

With a little practice, such an Ex. as this might be worked mentally, more quickly than it could be written.

APPLICATIONS OF PROPORTION.

RULE OF THREE.

The Articles marked thus (*) may be omitted by those who have not read Fractions.

132*. In articles (65) to (76) the subject of Proportion has been fully discussed: and our business now is, to shew how to work Exs., which are in reality only different forms of the question asked in (73), viz.: If three numbers be given, what fourth number must be chosen, such that the four, when taken in order, shall be proportionals?

133. The Rule of Three is so called because in the questions given under this head there are three quantities proposed. These three numbers, when placed in order for working an Ex., are called *terms*. In using the expression first, second, and third *terms*, we intend to indicate the manner in which these three given quantities are arranged. Thus, if I were to write 9, 8, 24, as first, second, and third *terms* respectively, in the sense here intended, I should have

$$\begin{array}{ccccc} \text{1st} & & \text{2nd} & & \text{3rd} \\ 9 & : & 8 & :: & 24 \end{array}$$

where the dots placed between the terms are signs which express the relation existing among these terms, so that with them and the fourth term a Proportion is formed. When any three terms are properly arranged at the commencement of an Ex. in Rule of Three, they are said to form a *Statement:* and the question is said to be stated. So long as only three terms are contained in a statement, the question is said to be one of *Simple Proportion:* but sometimes more than three quantities require to be so arranged: the question is then said to come under the head of *Compound Proportion*, or, as it is sometimes called, Double Rule of Three.

It is here evidently *assumed* that the three quantities, with the required fourth, will form a proportion. But to ascertain whether this is true, we must apply the tests given in (75) and (76); and if either of these is satisfied, the question may be worked according to the principles established there, and which are more fully drawn out in the next two Articles.

In the Appendix (Art. Proportion), three examples are produced, in two of which the existence of proportionality is shewn, which an ordinary learner could not assure himself to be true; and in the third, he might hastily assume that one of the usual forms of proportionality existed, where it is *not* true.

The pupil may work any ordinary question in the Rule of Three by the method given below; and it is to be observed that the process gives the solution without assuming the quantities to be proportionals; and, at the same time, by giving the same result as when they are so treated, shews that they may be rightly treated as proportionals, according to the common rule.

Ex. I. What is the coach fare for 130 miles, if it is £1 9*s*. 4*d*. for 85 miles?

The charge per mile is uniform, and the same in both parts of the question; hence, if

The fare for 85 miles be £1 9*s*. 4*d*.
,, 1 ,, will be $\frac{1}{85}$ of £1 9*s*. 4*d*.
∴ ,, 130 ,, ,, $\frac{130}{85}$ of £1 9*s*. 4*d*.

the same result as would be obtained by the ordinary method.

Ex. II. If when a peck of flour is sold for 1*s*. 4*d*., the penny loaf weighs 12 oz., how much will it weigh when wheat is 2*s*.?

At a rate of 16*d*. per peck, 1*d*. buys 12 oz.; and since, when the flour is cheaper, any sum, as 1*d*., will buy a proportionately larger amount of it,

∴ at a rate of 1*d*. per peck, 1*d*. will buy 16 times as much as when it was at 16*d*. per peck, *i. e.* at 1*d.* per peck, 1*d.* will buy 16×12 oz.: since also when the flour is dearer, the 1*d.* will buy proportionally less;

∴ at 24*d.* per peck, 1*d.* will buy $\frac{16 \times 12}{24}$ oz. = 8 oz.

Though the method here shewn dispenses with the necessity of using a statement, and is very valuable as a discipline to the pupil's mind, yet it requires greater dexterity in applying principles and in employing fractions, than many pupils possess who have advanced as far as Rule of Three; and this applies still more strongly when we come to Compound Proportion, and other Applications of Proportion, as we shall presently see; so that I think teachers will generally find it necessary to employ the common mode.

134*. Observing (73) and (74), we learn the following relations between the four terms forming any Proportion.

1st. That the third and fourth terms are of the same kind, *i.e.* that the third term must always be of the same nature as the one required.

2ndly. That the first and second must always be of the same kind; and that they must be reduced to the same denomination, if they be not already so expressed.

3rdly. That the fourth term is obtained by multiplying the third term by the fraction $\frac{\text{2nd}}{\text{1st}}$; or, if this operation be performed at two steps, we multiply by the second term, and divide by the first.

4thly. That if the second term be greater than the first, the fourth term will be greater than the third; but if the second term be less than the first, then the fourth term will be less than the third.

5thly. That since the fraction $\frac{\text{2nd}}{\text{1st}}$ is an abstract number, therefore the fourth term, which $= \frac{\text{2nd}}{\text{1st}} \times \text{3rd}$, is of the same denomination as that in which the third was expressed.

Collecting together these facts, we deduce the following Rule.

135. **Rule.** Find out the nature of the quantity sought by the question; *i. e.* of the required fourth term.

Of the three quantities given in the question, find that which is of the same nature as the fourth term, and take that quantity as the third term.

In order to place the other two quantities in their proper situations, inquire whether, from the nature of the question, the fourth term will be more or less than this third term: if *more*, make the *larger* of the two remaining quantities the middle or second term; but if *less*, make the *smaller* the middle term: the only remaining quantity must of course fill the first place.

If the first and second terms be not expressed in the same denomination, reduce them till they become so: and if the third term consist of several denominations, reduce it to the lowest name mentioned.

Then multiply the second and third terms together, and divide the product by the first: the quotient will be the answer or fourth term, expressed in the same denomination as that in which the third term was left.

If this quotient be expressed in too low a denomination, [as, for example, 1257 farthings, or 1836 dwts.,] let it be reduced to a higher denomination: [as £1 6*s.* 2¼*d.* and 7lb. 7oz. 16dwts.]

136. I will now proceed to work some Exs. which will illustrate the different varieties that may be expected under the head of Simple Proportion: and it will be found that the principal difficulty consists in arranging the three terms according to the directions prescribed by the Rule. This is especially the case, when the question is given in such a

shape, that the three terms cannot be *immediately* obtained from the question as it stands. I will explain this more fully as I go through the various kinds of Exs. When the *Statement* is once obtained, the remainder of the work consists merely of Multiplication, Division, and Reduction.

I recommend a pupil to work every question that I have worked, so that he may the better see the correctness and ascertain the object of the successive operations in any Example.

Ex. I. If 12 yards of cloth cost £19, what will 8 yards cost?

Here I see that the required fourth term will be money; I therefore place the £19 in the third term. Also, the fourth term, which is to be the price of 8 yards, will be *less* than the third term, which is the price of 12 yards; therefore, according to the Rule, I place the *smaller* of the two remaining terms, *i. e.* the 8 yards, in the middle, and the 12 yards in the first place.

yds. yds. £
12 : 8 :: 19
8
12)152
£12 13*s.* 4*d.*

The first and second terms are already in the same name, and the third term contains but one denomination, therefore no reduction is required. I now multiply the second and third terms together, and divide by the first: the answer is £12 13*s.* 4*d.* And this fourth term and the other three terms form the following proportion:—

12 yds. : 8 yds. :: £19 : £12 13*s.* 4*d.*

or in words, 12 yards are to 8 yards, as £19 are to £12 13*s.* 4*d.*

Ex. II. If 17 cwt. 3 qrs. and 14 lbs. cost £8 18*s.* 9*d.*, how much may be bought for £5 12*s.* 6*d.* at the same rate?

The fourth term will evidently be expressed in weight; therefore I put 17 cwt. 3 qrs. 14 lbs. in the third term. I now ask this question: "If the quantity in the third term can be obtained for £8 18*s.* 9*d.*, will more or less be bought for the £5 12*s.* 6*d.*?" evidently *less;* therefore I place the *less* of the two prices in the middle, and the remaining one first. The first and second terms are not expressed in any single denomination; I therefore reduce them to threepences, which is the highest denomination to which they can both be reduced. Also, since the third term consists of more denominations than one, I reduce it to the lowest denomination entioned, viz. lbs. After I have multiplied the second and third terms gether, and divided by the first, the quotient is 1260, which consists of s., because the third term was expressed in lbs. This quotient, when educed to higher denominations, becomes 11 cwt. 1qr.

```
 £  s.  d.     £  s.  d.     cwts. qrs. lbs.
 8  18  9  :   5  12  6  ::   17    3   14
 20            20              4
 ---           ---            --
 178           112            71
   4             4            28
 ---           ---            ---
 715           450           582
 ===           ===          142
                            ----
                            2002
                             450
                           ------
                           100100
                           8008
                   715) 900900 (1260 lbs.
                        715
                        ----
                        1859
                        1430        28 { 4) 1260
                        ----           {   -----
                         4290          { 7) 315
                         4290              ----
                         ----            4) 45
                            0          -------------
                         ====           11 cwt. 1 qr.
                                        =============
```

I will now give a few Exs. in which the difficulty consists in preparing the question for being stated; but I shall merely show how to overcome the difficulty, and leave the question to be worked out as in the former Exs.

Ex. III. A bankrupt's effects amounted to £980 10*s.*, and he paid his creditors 13*s.* 4*d.* in the pound; what was the amount of his debts?

At first sight there appear to be only two terms in this question, but the £1 furnishes another term. Now all the three quantities are money; but by reading the question thus: "If 13*s.* 4*d.* be paid for a debt of £1, what debt will be paid by £980 10*s.*?" I learn that 13*s.* 4*d.* and £980 10*s.* are money *paid*, and the £1 is money *owed*, or *debt;* and since the fourth term is *debt*, I place the £1 in the third term. Also, the debts which are paid by £980 10*s.* are of course more than this third term; therefore I place the larger term, £980 10*s.*, in the middle, and 13*s.* 4*d.* in the first place. The statement will then be 13*s.* 4*d.* : £980 10*s.* :: 1£. The fourth term will be found to be £1470 15*s.*

137*. The four terms, arranged as a proportion, will be

13*s.* 4*d.* : £980 10*s.* :: £1 : £1470 15*s.*

or, as two equal ratios—

$$\frac{13s.\ 4d.}{£980\ 10s.} = \frac{£1}{£1470\ 15s.}$$

which, expressed in words, indicates that the payment, 13*s.* 4*d.*, is the same portion of the whole payment £980 10*s.* that the debt of £1 is of the whole debt £1470 15*s.*

138. **Ex. IV.** How much may a person spend in 73 days, if he wishes to lay by every year 50 guineas out of an income of £450?

Since I wish to know how much may be spent in 73 days, I must ascertain how much is spent in one year: I therefore subtract the 50 guineas, which are saved, from the whole income of £450: the remainder is £397 10*s.*, and the question now becomes—"If in 365 days I spend £397 10*s.*, how much may I spend in 73 days?" The statement will be

$$\begin{array}{ccccr} & & & & £\quad s. \\ & & & & 450\quad 0 \\ \text{days.} & & \text{days.} & & \underline{52\quad 10} \\ 365 & : & 73 & :: & 397\quad 10 \end{array}$$

and the required sum will be found to be £79 10*s.*

Ex. V. If the sixpenny loaf weigh 3lbs. when wheat is 6*s.* per bushel, what should it weigh when wheat is 6*s.* 9*d.* per bushel?

This seems a very simple question, but most pupils make an error in the statement; for, after correctly placing the 3 lbs. in the third term,—when they ask the question, as to whether the answer will be more or less than this term, they reason that if the loaf weigh 3lbs. when wheat is at 6*s.*, it will weigh more when wheat is at 6*s.* 9*d.*; whereas, since the wheat is dearer, we ought to have a less weight of bread for the same money: and the statement will be

6*s.* 9*d.* : 6*s.* :: 3lbs.

Where questions are met with involving the necessity of finding the area of a surface, or the volume of a solid, we must refer to Art. Duodecimals; or, if the dimensions are expressed in any single denomination, we may refer to Arts. (22) and (28) in Part I. of the Arithmetic.

139*. I will now by a few Exs. illustrate the use of Frac^{ns}.

Taking the statement already made in Ex. III., I have

$$\text{Fourth term} = \frac{\text{2nd}}{\text{1st}} \times \text{3rd} = \frac{£980\ 10s.}{13s.\ 4d.} \times £1$$

$$\left(\begin{array}{l}\text{when first and second are re-}\\ \text{duced to a fractional part}\\ \text{of a £.}\end{array}\right) = \frac{980\frac{1}{2}£}{\frac{2}{3}£} \times £1$$

$$= \left(\frac{1961}{2} \times 1 \times \frac{3}{2}\right)£$$

$$= \frac{5883}{4}£ = 1470\tfrac{3}{4}£$$

$$= £1470\ 15s.$$

Again, in Ex. V.

$$\text{Fourth term} = \frac{6s.}{6\frac{3}{4}s.} \times 3\text{lbs.} = \frac{6}{\frac{27}{4}} \times 3\text{lbs.} = \frac{\overset{2}{\cancel{6}} \times \cancel{3} \times 4}{\underset{\underset{3}{\cancel{9}}}{\cancel{27}}}\text{lbs.} = \frac{8}{3}\text{lbs.}$$

$$= 2\tfrac{2}{3}\text{lbs.}$$

Ex. VI. If $4\frac{5}{9}$oz. avoirdupois cost $\frac{31}{32}s.$, what will $8\frac{13}{24}$lbs. cost?

$$4\tfrac{5}{9}\text{oz.} : 8\tfrac{13}{24}\text{lbs.} :: 8\tfrac{31}{32}s.$$

$$\text{Fourth term} = \frac{8\frac{13}{24}\text{lbs.}}{4\frac{5}{9}\text{oz.}} \times 8\tfrac{31}{32}s. = \frac{\frac{205}{24} \times 16\text{oz.}}{\frac{41}{9}\text{oz.}} \times \frac{287}{32}s.$$

$$= \left(\frac{\overset{5}{\cancel{205}}}{\underset{8}{\cancel{24}}} \times \frac{\cancel{16}}{1} \times \frac{287}{\underset{2}{\cancel{32}}} \times \frac{\overset{3}{\cancel{9}}}{\cancel{41}}\right)s. = \frac{15 \times 287}{16}s.$$

$$= \frac{4305}{16}s. = 269\tfrac{1}{16}s.$$

$$= £13\ 9s.\ 0\tfrac{3}{4}d.$$

Questions wherein fractional quantities occur in the terms may also be worked by the use of Decimals; but this is not generally an advantageous method of solving them, especially when some of the decimals are circulating. See Ex. VI. in (119).

Exs. 46.

1. If 24 men earn £36, what sum will 42 men earn at the same rate?

2. In how many days will 5 guineas be spent, at the rate of 7 shillings in 3 days?

3. A wall containing 372 square feet is paid for at the rate of 1*s.* 10½*d.* per square yard; find the cost of the whole.

4. How many yards of paper, 27 inches wide, will hang a room 54 feet round and 10 feet high?

5. If an acre is 220 yards in length, and 22 in breadth, what must be the length when the breadth is 27½ feet?

6. If I can buy 15¾ yds. of cloth for 10 guineas, how much can I buy at the same rate for £283 17*s.* 6*d.*?

7. An income of £150 pays a tax of £4 7*s.* 6*d.*; what will be the tax upon £586 1*s.*?

8. If the carriage of 15½ cwt. for 56 miles come to 10*s.* 6*d.*, how much can I have carried 72 miles for the same money?

9. The carriage of 4 cwt. for 72 miles cost 15*s.* 9*d.*; how many lbs. can be carried 13¾ miles for the same sum?

10. A field of 16 acres produces 440 bushels of wheat: how much is that upon every 22 square yards?

11. A creditor agreeing to receive £51 for a debt, finds that he has been paid at the rate of 12*s.* 9*d.* in the pound; how much was the debt?

12. Three men who weave at the rate of 5¼ yds. per day, finish 119 yds. in a certain time: at what rate per day must they weave, who finish 85 yds. in the same time?

13. If 15 men eat 35 shillings' worth of bread in a certain time, when wheat is 12*s.* per bushel; how much may they eat at the same cost when wheat is 7*s.* per bushel?

14. A rate of 2*s.* 9*d.* in the pound produces £352; what is the rental of the parish?

15. If an income-tax of 10*d.* in the pound amounts to 2½ millions, what must be the poundage in order to produce £3,375,000?

16. A man buys 148 yds. at 2*s.* 7½*d.* per yd.; at what price must he sell it to gain £12 12*s.* 10*d.* by the whole?

17. The penny loaf weighs 10 oz. when wheat is 5*s.* a bushel; how much will it weigh when wheat is 6*s.* 3*d.*?

18. The rental of a parish is £5626 10*s.*; and the assessment for the poor-rates is £405 13*s.* 4*d.*; how much will be the rate on £45 12*s.* 9*d.*?

19. If the net income of an estate after paying all taxes be £534 15*s.*, and the gross income be £570 8*s.*; how much in the pound did the taxes amount to?

20. The chain for measuring land is 4 perches in length, and is divided into 100 links; what is the length of a wall in feet, which measures 1550 links?

21. If 1 lb. of gold, and 1 oz. of alloy, can be coined into 44½ guineas; find the value of 5 oz. of pure gold, considering the alloy of no value.

22. I spend 12 guineas in 35 days, and save £100 a year: what must I earn in the year?

23. A piece of gold at £3 17*s.* 10½*d.* per oz. is worth £150: what will be the worth of a piece of silver of equal weight, at 54*s.* 6*d.* per lb.?

24. Two floors are equal in size, one is 35 feet long, and 25 feet broad; the other is 40 feet long: what is its breadth?

25. The weights of gold and of water are as 19¼ and 1; find what number of solid inches of gold is equal in weight to 17½ cub. ft. of water?

26. A dollar is to a crown as 111 : 120; how many dollars are equal in value to £250?

27. A clock which gains 7½ minutes in 24 hours, is 14 minutes fast at Monday midnight; what time will it indicate at 6 o'clock in the evening of the following Thursday?

28. A person owes £1537 3*s.* 4*d.*; but can pay only £960 14*s.* 7*d.*: what will be the dividend, and how much shall I receive for a debt of £276 11*s.* 6*d.*?

29. The shadow of a stick 3ft. 6in. long is 2ft. 9in.: what is the height of a tree which at the same time throws a shadow of 154 feet? (See App. Art. Proportion.)

30. An income of £3827 12*s.* 6*d.* is taxed at the rate of 7*d.* in the pound; how much clear income will remain?

31. Bought 236 gallons of oil for £111 6*s.* 8*d.*; what profit will be made by selling it at the rate of 8*s.* 6*d.* for 3 quarts?

32. Paid £45 10*s.* for a hogshead of rum; how much water must be added, to be able to sell it without loss or gain at 11*s.* 6*d.* per gallon?

33. Out of an income of £312 10*s.* a year, the expenses are £55 in 146 days; in what time will 1000 guineas be saved?

34. If 90 English degrees correspond to 100 French degrees, how many French degrees are there in 36·45 English?

35. What must be the breadth of a piece of ground which is 14⅝ yds. long, so that it may be as large as a piece 40½ yds. long and 4¾ broad?

36. The 6*d.* loaf weighs 3¼ lbs., when wheat is 50*s.* a quarter; what will it weigh when wheat is 40*s.* 3*d.* a quarter?

37. Given that the velocity of a falling body is proportional to the time during which it falls; find the time of descent of a body having acquired a velocity of 1000 feet, supposing that the velocity obtained in 2½ seconds is 80·5 feet.

38. The lengths of the arms of a lever are inversely proportional to the weights at the extremities of the arms; if the lengths of the arms be 3 feet and 2¼ inches, what must be the weight at the longer arm to balance 20 lbs. at the shorter end?

39. A bankrupt paid £1520 to his creditors; £205 of his debts were paid in full; and his assets were to his remaining debts as 3 : 8; find the amount that he owed.

40. What can a man save per annum, who out of an income of £500 gives away $\frac{1}{15}$th, pays 7*d*. in the pound income-tax, and spends £$16\frac{2}{3}$ in 3 weeks?

COMPOUND PROPORTION.

140. It was observed in (133) that a question was classed under the head of Compound Proportion, when there were more than three quantities which required to appear in the statement. The following is an Example.

Ex. I. If 12 yards of cloth, 3 quarters wide, cost £19, what will be the cost of 8 yards, 5 quarters wide?

If the width of the two pieces of cloth were the same, we should take no account of this width, whatever it might be: and the question would then become Ex. I. in (136), or one of Simple Proportion. Referring to that Ex. we have the statement

$$12 \text{ yards} : 8 \text{ yards} :: £19 \qquad \text{(B)}$$

and the cost of the new piece $= \frac{8 \text{ yds.}}{12 \text{ yds.}} \times £19 = \frac{8}{12} \times 19£.$

We will now take into account the two breadths, 3 qrs., and 5 qrs., and put the following question:—"If a piece of cloth cost $\frac{8 \times 19}{12}£$, when 3 quarters wide, what will it cost when the width is 5 quarters?" The statement would be

$$3 \text{ qrs.} : 5 \text{ qrs.} :: \frac{8 \times 19}{12}£ \qquad \text{(C)}$$

$$\text{and the fourth term} = \frac{8 \times 19}{12}£ \times \frac{5 \text{ qrs.}}{3 \text{ qrs.}} = \frac{8 \times 19 \times 5}{12 \times 3}£.$$

Now, if the whole of statement (B), and the first and second terms in (C), be converted into one statement, as follows:—

$$\begin{matrix} 12 \text{ yds.} \\ 3 \text{ qrs.} \end{matrix} : \begin{matrix} 8 \text{ yds.} \\ 5 \text{ qrs.} \end{matrix} :: £19 \qquad \text{(D)}$$

and we take the product of the two quantities which stand first, as our first term, and the product of the two in the middle, as our second term, we shall have

$$\text{the fourth term} = \frac{8\text{ yds.}}{12\text{ yds.}} \times \frac{5\text{ qrs.}}{3\text{ qrs.}} \times 19£ = \frac{8 \times 5 \times 19}{12 \times 3}£,$$

which is precisely the same as was obtained from the two successive statements: hence such a statement as (D) will produce a correct result. Also, in forming this statement, independently of (B) and (C), I select for the third term that which is similar to the required fourth term, as in Simple Proportion: and in placing the remaining terms, I take each pair separately, and ask the usual question with the third term, as to whether the answer will be more or less than this term; and I arrange this pair precisely as though they were the only two terms which I had to consider. However many pairs of terms occur in the question, they must all be treated in like manner; for the same proof that has shewn how to combine the first and second statements, will shew how to combine the third with the result of the first pair.

I will work another Ex. and mention the mental operations which must be performed, in order to enable me to place each pair of terms correctly.

Ex. II. A field, 300 yards long and 280 broad, was ploughed by six horses in two days of eight hours each; how many horses will plough a piece of ground 500 yards long, and 315 broad, in three days of ten hours each?

$$\left.\begin{array}{c} 300 \text{ yards} \\ 280 \text{ yards} \\ 3 \text{ days} \\ 10 \text{ hours} \end{array}\right\} : \left\{\begin{array}{c} 500 \text{ yards} \\ 315 \text{ yards} \\ 2 \text{ days} \\ 8 \text{ hours} \end{array}\right. :: 6 \text{ horses}$$

The fourth term will be horses: I therefore place the six horses in the third term. Also, the new field is longer than the one which required six horses; hence, considering the effect of this pair of terms alone, it will require more horses, and I place the larger term, 500, in the middle: so, also, the second field is broader than the first, and therefore will take more horses, and I place 315 in the middle. Again, the first field was ploughed in two days, but the new one in three days; hence, since the time is longer, we shall, so far as this pair of terms is concerned, require fewer horses; and the smaller term, two, is to be in the middle. Also, in ploughing the first field, the days were eight hours, but for the second field they are ten hours; hence, with this extra time, fewer horses will be required, and I place the eight in the second place. The whole statement is as annexed; and

$$\text{the fourth term}=\frac{\overset{\cancel{5}}{\cancel{1}}\cancel{5}\cancel{0}\cancel{0}\times\overset{\cancel{9}\,3}{\cancel{3}\cancel{1}\cancel{5}}\times2\times\cancel{8}\times\overset{\cancel{2}}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{3}}\cancel{0}\cancel{0}\times\underset{\cancel{3}\cancel{5}}{\cancel{2}\cancel{8}\cancel{0}}\times\cancel{8}\times\underset{\cancel{2}}{\cancel{1}\cancel{0}}}\text{ horses}$$

$$=3\times2\text{ horses}=6\text{ horses};$$

precisely the same number as before: *i. e.* the increased size of the field, and the increased length of time allowed for the work, are so balanced, that the same number of horses as before is sufficient.

In arranging the several pairs of terms in the statement, I seem to be trying at one time to obtain a smaller term, and at another time a larger term than the third; and it is true that some conditions of the question tend to make the fourth term less than this third, and some to make it more. Each pair will produce its own effect in increasing or diminishing the required term; and we shall therefore find the result more or less than the third, according as the conditions in the question which would make it more, predominate, or not, over those which would make it less; *i. e.* according as the product of all the terms in the second place is more or less than the product of all in the first place.

It must be carefully observed, that if any pair of corresponding terms be not expressed in the same denomination, they must be so reduced, just as in Simple Proportion, before we commence forming the fraction which will give the fourth term.

According to the method of (133), without assuming the proportionality of the quantities involved, the work will be as follows.

In Ex. I. let A be the number of pounds sterling required, then

12 × 3 represents the area of cloth which cost £19

and 8 × 5 „ „ „ £A

$$\therefore \frac{12 \times 3}{19} \text{ represents the area costing £1}$$

$$\text{and } \frac{8 \times 5}{A} \quad ,, \quad ,, \quad ,, \quad \text{£1}$$

and since £1 will always buy the same quantity, the quality being supposed the same, therefore, inverting these equal fractions,

$$\frac{A}{8 \times 5} = \frac{19}{12 \times 3} \text{ or, } A = \frac{8 \times 5 \times 19}{12 \times 3}, \text{ as before.}$$

In Ex. II. let A be the number of horses;

Now, 6 horses can do 300×280 sq. yds. in 2×8 hrs.

$$\text{or, } \frac{300 \times 280}{2 \times 8} \text{ sq. yds. in 1 hr.}$$

$$\therefore \text{1 horse can do } \frac{300 \times 280}{2 \times 8 \times 6} \text{ sq. yds. in 1 hr.} \qquad \text{(I)}$$

$$\text{So, also, } A \text{ horses can do } \frac{500 \times 315}{3 \times 10} \text{ sq. yds. in 1 hr.}$$

$$\therefore \text{1 horse } ,, \quad \frac{500 \times 315}{3 \times 10 \times A} \text{ sq. yds.} \qquad \text{(II)}$$

$$\text{(I) and (II) are equal; } \therefore \frac{A \times 3 \times 10}{500 \times 315} = \frac{2 \times 8 \times 6}{300 \times 280}$$

$$\text{or, } A = \frac{500 \times 315 \times 2 \times 8 \times 6}{300 \times 280 \times 3 \times 10}, \text{ as before.}$$

Exs. 47.

1. If 10 men can dig 30 yds. of earth in 8 days, how many yards can be dug by 20 men in 4 days?

2. If £300 gain £10 in a year, in what time will £900 gain £175 10*s.*?

3. Three boats take 6000 herrings in 8 days; in how many days will 450 boats take 20,000 barrels, each containing 700 herrings?

4. I borrow £175 10*s.* for 10 months, when money is worth 5 per cent.; how much must I lend in return for 12 months, when money is worth $3\frac{1}{2}$ per cent.?

5. If five men can reap a field whose length is 800 feet and breadth 700, in $3\frac{1}{2}$ days of 14 hours each; in how many days of 12 hours each can seven men reap a field whose length is 1800 feet, and breadth 960 feet?

6. The papering of a room $10\frac{1}{2}$ feet high, and 20 yards round, cost £1 2*s.* 6*d.*; what will be the cost of papering another room 9 feet high, and 63 feet round?

7. If I pay 1*s.* 3*d.* for 6lb. 14oz. of bread, when wheat is 4*s.* 9*d.*

per bushel, what must I pay for 23lb. 12oz., when wheat is *5s. 5d.* per bushel?

8. A printing machine turns out 37,260 sheets in a day, running 12½ hours; if its speed be increased in the ratio of 4 to 3, how many sheets will be wrought in 7½ hours?

9. A carriage wheel, the circumference of which is 16½ feet, and which makes 45 revolutions per minute, goes 275 miles in a certain time; how many revolutions per minute must a wheel make, to perform 385 miles in the same time, the circumference of the latter wheel being 19¼ feet?

10. A field of 12 acres having 120 stalks to each square yard, and 70 grains to each stalk, produces wheat to the value of £96 16*s.*: what will be the worth of the produce of 800 square yards, having 175 stalks to the square yard, and 45 grains to each stalk?

11. An iron beam 16 ft. long, 2¼ ft. broad, and 8 in. thick, weighs 1280 lbs.: what must be the length of a beam whose breadth is 3¼ ft., thickness 7½ in., and weight 2028 lbs.?

12. If a wheel which revolves at the rate of 470 times in 8 minutes, make 50 revolutions in a certain time; how many revolutions will another wheel make in the same period, at the rate of 360 revolutions in 7 minutes?

13. If 15 men eat 13*s.* worth of bread in 7 days, when wheat is 12*s.* per bushel; what should be the price so that 10 men should be furnished for 12¼ days at the same cost?

14. A hay-field which has 2½ tons to the acre is mown by 20 men in 6 days working 8 hours a day; what number of hours per day must 13 men work for 8 days upon a field which has 3¼ tons to an acre?

15. The circumferences of the smaller and larger wheels of a carriage are in the ratio of 5 to 6. Let the carriage move in a ring, so that the circumferences of the circles described by the inner and outer wheels shall be as 7 : 8. Given that the inner large wheel makes 800 revolutions in describing ⅓ of its path, find the number of revolutions made by the small outer wheel, while describing ⅓ of its path.

16. If the price of 100 bricks, of which the length, breadth, and thickness are 16, 8, and 10 respectively, be *5s. 4d.*; what will be the price of 9760 bricks, which are one-fourth greater in every dimension?

17. If 5 steam-engines of 9-horse power (when employed 3 days a week, and 10 hours a day) raise in one week through a certain altitude 25 three-bushel sacks of wheat, weighing 60lbs. a bushel; in what time will [illegible] engines of 8-horse power (when employed 5 days in the week, and

9 hours a day) raise through 15 times the former altitude, 75 two-bushel sacks of wheat, weighing 63 lbs. a bushel?

18. Three fire-engines, each having 4 pipes, 3 square inches in section, are worked at the rate of 20 strokes in 3 minutes, and discharge 4680 gallons of water in 16 minutes; how many engines, each having 3 pipes, 5 square inches in section, and worked at the rate of 17 strokes in $2\frac{1}{2}$ minutes, will discharge 20,000 gallons in half an hour?

INTEREST.

141. INTEREST is the payment made for the use of money for any time, and is generally reckoned at so many pounds a-year for £100 lent; or as it is commonly called, so many pounds per cent. For instance, if £5 be the interest of £100 lent for a year, we should say that the money is lent at the rate of five per cent. per annum.

The sum lent is called the *Principal;* the interest of £100 for one year the *Rate;* and the sum lent, together with its interest, for any length of time, is called the *Amount.*

When interest is paid only upon the sum originally lent it is called *Simple* Interest; but when at the end of any time agreed upon, as for instance a year, the interest is added to the principal, so that this amount forms the principal for the next year; and a similar addition is made at the end of every such period, then it is termed *Compound* Interest.

142. The questions which occur in this Rule are merely Examples of Proportion.

Ex. I. Find the Simple Interest of £382 10*s.* for one year at 5 per cent.; in other words—If the principal £100 give £5 interest, what interest will be derived from the principal £382 10*s.*? the statement will evidently be

£100 : £382 10*s.* :: £5.

Also, it will be found that every Ex. in Simple Interest will furnish a similar statement, in which we observe that the first term is £100, the second is the principal, and the third is the rate.

We now multiply the second term by the 5, and divide by the 100; *i. e.* we multiply by the rate and divide by 100. And in these Exs. we do not, as usual in the Rule of Three, reduce the first and second terms to the lowest denomination expressed in either of them, but multiply by 5 and divide by 100, as in Compound Multiplication and Division. The sum, when worked in the usual form, stands thus:

$$\begin{array}{rrl}
\pounds 100 & : \ \pounds 382\ 10s. & :: \ \pounds 5 \\
 & \underline{5} & \\
1{,}00) & 19{,}12\ 10 & \\
 & \underline{20} & \\
 & 2{,}50 & \\
 & \underline{12} & \\
 & \underline{\underline{6{,}00}} & \textit{Answer.}\ \pounds 19\ 2s.\ 6d.
\end{array}$$

Here the division by 100 is performed by cutting off two ciphers at the end of the divisor, and two figures at the end of the dividend: and the remainder after each division is reduced, as in Compound Long Division. I have written the 100 as a divisor, but in practice it is omitted.

143. Since in (142) we multiplied by 5, and divided by 100, therefore we might at once have multiplied by the fraction $\frac{5}{100}$, or $\frac{1}{20}$: that is, the operation of finding the interest might have been performed mentally by taking $\frac{1}{20}$th part of the principal; similarly, $\frac{1}{40}$th, for $2\frac{1}{2}$ per cent.; $\frac{1}{10}$th, for 10 per cent. &c.; and this short method is generally used when the rate is an aliquot part of £100, but not otherwise.

144. If the interest for any number of years is required, multiply the interest for one year by the number of years. If for any number of months, aliquot parts of the interest

for one year may be taken, as in Practice: but if for any number of weeks or days, it should be generally found by Proportion.

Ex. II. Find the interest of £175 for three years and 135 days, at five per cent.

The interest of £175 for one year is £8 15*s*., and for three years is 3 × (£8 15*s*.) = £26 5*s*. Now, to find what is the proportionate amount of interest for 135 days, we have this statement;

$$\begin{array}{ccccc} \text{days.} & & \text{days.} & & \text{£ s.} \\ 365 & : & 135 & :: & 8\ 15 \end{array}$$

and the answer is £3 4*s*. $8\frac{52}{73}$*d*.: therefore the whole interest = £26 5*s*. + £3 4*s*. $8\frac{52}{73}$*d*. = £29 9*s*. $8\frac{52}{73}$*d*.

Exs. of this kind, involving Simple Interest for years and days, may also be worked as follows:

Since we obtain Simple Interest for 1 year by multiplying by the rate per cent., and dividing by 100,

$$\therefore \text{in this case, S. Int}^{t} \text{ for 1 year} = 175 \times \frac{5}{100}\,\text{£} \qquad \text{(E)}$$

and expressing the 135 days as a fractional part of a year, and multiplying the interest of 1 year by the number of years, viz. $3\frac{135}{365}$, or $3\frac{27}{73}$,

$$\therefore \text{whole interest} = 175 \times \frac{5}{100} \times 3\frac{27}{73} = 175 \times \frac{5}{100} \times \frac{246}{73}$$

$$= \frac{215250}{7300} = \text{£}29\ 9s.\ 8\frac{52}{73}d.$$

Obs. I did not reduce the fr^{n} $\frac{5}{100}$ in (E) to lower terms, because the den^{r} is generally more simple, when the 100 is left uncancelled.

Under the head of Simple Interest may be included all questions generally classed under the heads of Commission, Brokerage, and Insurance; for all such quantities are calculated at a fixed rate for every £100.

Exs. 48.

Find the Interest of	£	*s.*	*d.*		*yr.*		£		Find the Amount of	£	*s.*	*d.*		*yrs.*		£	
1.	324	0	0	for	1	at	5	p. c.	4.	1025	0	0	for	3	at	2¾	p. c.
2.	475	10	6	,,	1	,,	4	,,	5.	1750	9	0	,,	1½	,,	3½	,,
3.	875	12	3	,,	1	,,	3¼	,,	6.	1827	18	9	,,	6½	,,	2¼	,,

Interest of

£ s. d. y. w. £
7. 540 17 6 for 1 5 at 4 p. c.
8. 1845 guineas ,, 3 10 ,, 3 ,,

Amount of

£ s. d. y. w. £
9. 237 10 0 for 4 11 at 5¼ p.c.
10. 11428 0 0 ,, 9 7 ,, 1¾ ,,

Interest of

£ s. d. y. w. £
11. 755 6 8 for 1 15 at 2¾ p.c.
12. 935 13 4 ,, 3 39 ,, 4½ p.c.

Amount of

£ s. d. y. w. £
13. 1875 5 0 for 3 45 at 4¼ p.c.*
14. 2000 0 0 ,, 11 30 ,, 3½ ,,

Interest of

£ s. d. y. d. £
15. 1440 15 0 for 1 73 at 5 p.c.
16. 2500 0 0 ,, 3 90 ,, 4 ,,

Amount of

£ s. d. y. d. £
17. 1175 2 6 for 4 300 at 3¼ p.c.
18. 990 11 0 ,, 7 150 ,, 6 ,,

145. As an Ex. in Compound Interest we may take the following question :—

Ex. III. What is the amount of £350 in three years at five per cent. Compound Interest?

Putting down only the results of the three operations, we have

	£	s.	d.
Principal of first year	350	0	0
Interest of first year	17	10	0
Amount of first year, and principal of second year .	367	10	0
Interest of second year............	18	7	6
Amount of second year, and principal of third year	385	17	6
Interest of third year	19	5	10½
Amount at the end of third year............	405	3	4½

This is the amount; if the interest alone is required, we subtract the principal £350, from this amount; the remainder is £55 3*s.* 4½*d.*

The above results can be obtained mentally when the per centage is an aliquot part of 100, as 2½, 5, 10, 20, (143); or when it can be easily made up of such numbers, as 21 p. c. in (152), Ex. III.; but if any other be taken, the work must be performed at length, as in (142).

Exs. 49. Find, at Compound Interest,

The Interest of

£ s. d. yrs. £
1. 1500 0 0 for 4 at 5 p. c.
2. 354 13 6 ,, 2½ ,, 4½ ,,
3. 1820 15 0 ,, 3¼ ,, 2½ ,,

The Amount of

£ s. d. £
4. 750 0 0 for 6*y.* at..... 5¼ p. c.
5. 2025 0 0 ,, 3*y.* 150*d.* at 5 ,,
6. 1825 11 6 ,, 4*y.* 7*mo.* ,, 3 ,,

---

* In this and the following Exs. the fractional parts of a penny have been neglected when finding the interest for the weeks.

In Compound Interest, when the int^t is paid half-yearly, or quarterly, the result is different from that obtained when it is paid yearly, because the principal is sooner increased by the addition of interest; and therefore Comp^d Int^t begins sooner to accrue. In all cases, the int^t must be added to the principal as soon as it is due.

146. Questions may be found in Interest which involve some little difficulty, because there do not appear at once three terms out of which to form a statement. And most pupils will find that in any difficult question involving the application of Proportion, as in Profit and Loss, Stocks, &c., they cannot succeed in thoroughly comprehending it, without placing it in a plain Rule of Three form.

Hitherto we have been finding the *Interest*, when the other three quantities, *Principal*, *Rate*, and *Time* were known; the following Exs. shew how to find *any one* out of the above four quantities, when the remaining three are known.

Ex. IV. In what time will £75 12*s*. 6*d*. amount to £99 16*s*. 6*d*. at four per cent. per annum?

Here the interest gained is found by subtracting the principal, £75 12*s*. 6*d*., from the amount, £99 16*s*. 6*d*., and it = £24 4*s*. Also, the interest of £75 12*s*. 6*d*. for one year at 4 per cent. is £3 0*s*. 6*d*. Hence I have this question:

If £75 12*s*. 6*d*. produce £3 0*s*. 6*d*. in one year, in what time will it produce £24 4*s*.? The statement is

£3 0*s*. 6*d*. : £24 4*s*. :: 1 year,

and the fourth term will be found to be 8 years.

Ex. V. What sum will amount to £104 2*s*. 6*d*. in four years, at $4\frac{1}{2}$ per cent. simple interest?

I must here inquire what £100 would amount to in four years, at $4\frac{1}{2}$ per cent.; the answer is £118. The question is now, therefore,

If £100 become £118, what sum will in the same time amount to £104 2*s*. 6*d*.? The statement is

£118 : £104 2*s*. 6*d*. :: £100

and the required fourth term is £88 4*s*. $9\frac{57}{59}$*d*.

Ex. VI. At what rate per cent. will £152 10*s*. amount to £191 7*s*. 9*d*. in six years?

I have in this Ex. to find the interest of £100 for one year. Now the interest gained by £152 10*s.* in six years is (£191 7*s.* 9*d.* − £152 10*s.*), or £38 17*s.* 9*d.*, or in one year £6 9*s.* 7½*d.*: hence the question now is,

If £152 10*s.* gain £6 9*s.* 7½*d.*, what will £100 obtain? The statement will be

£152 10*s.* : £100 :: £6 9*s.* 7½*d.*

Obs. These, and all future Exs., as in Profit and Loss, Partnership, &c., of which I put down only the steps, should be worked out fully by a pupil, and in all cases, where possible, fractionally. Thus in the last Ex., since £6 9*s.* 7½*d.* = 129*s.* 7½*d.* = 129⅝*s.*,

$$\text{the 4th term} = \frac{£100}{£152\frac{1}{2}} \times 129\tfrac{5}{8}s. = \frac{100}{\frac{305}{2}} \times \frac{1037}{8}s. = 100 \times \frac{2}{305} \times \frac{1037}{8}s.$$

$$= 85s. = £4\ 5s.$$

i. e. the rate per cent. is 4¼.

Exs. 50.

1. In what time will £62 amount to £71 6*s.* at 5 per cent. per annum*?

2. In what time will £1215 15*s.* amount to £1291 14*s.* 8¼*d.* at the rate of 2½ per cent.?

3. For how many years must I put out £987 12*s.* to interest at 4¼ per cent., in order that I may receive £1197 9*s.* 3¾*d.*?

4. What sum of money will in 1 year amount to £108 13*s.* 6*d.* at 3½ per cent.?

5. Required the principal which will in 3 yrs. amount to £136 2*s.* 1½*d.*, at 2¼ per cent.

6. Find what sum will produce interest amounting to £330 15*s.* in 7 yrs. at 4½ per cent.

7. At what rate of interest will £95 15*s.* amount to £112 10*s.* 1½*d.* in 5 yrs.?

8. What must be the per centage in order that £1175 may become £1637 13*s.* 1½*d.* in 7½ yrs.?

9. Required the rate per cent. at which 1000 guineas will gain £590 12*s.* 6*d.* in 12½ yrs.

10. Find the rate per cent. at which any sum of money will double itself in 8 yrs.

11. What sum lent at 5 per cent. Compound Interest will in 3 years amount to £358 17*s.* $3\frac{3}{10}$*d.*?

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* Simple Interest is always implied, unless the contrary be expressed.

12. Find the difference between the Simple and Compound Interest of £416 13*s.* 4*d.* for 2 yrs. at 2¼ per cent.

13. What is the commission on £20500 at 3¼ per cent.?

14. Find the premium on a policy of life insurance for £2500 at £5 17*s.* 9*d.* per cent.

15. What annual premium must a farmer pay on stock valued at £370 10*s.*, at 2*s.* 3*d.* per cent.?

16. In what time will £818 18*s.* 4*d.* amount to £1064 11*s.* 10*d.* at 3¾ per cent.?

17. At what rate of interest will £782 10*s.* amount to £1245 5*s.* in 10 yrs.?

18. If the interest on £130 15*s.* 10*d.* for 10 days be 3*s.* 7*d.*, how much is that per cent. per annum?

147. Questions concerning Annuities, Leases, and Reversions involve applications of Interest; but they generally require for their solution either algebraical expressions or tables derived from these. The following is however a simple example of the kind.

Ex. VI. What is the amount of an annuity of £50 left unpaid for 5 yrs., allowing Compound Interest at 4 per cent. per annum?

I write down merely the outlines of the work, and neglect all sums below 1*d.*, as the fractions obtained after two or three divisions become exceedingly heavy.

	£	*s.*	*d.*
Amount due at the end of first year	50	0	0
Interest due at the end of second year	2	0	0
Add £50, due at the end of second year	50	0	0
Principal of third year	102	0	0
Interest at the end of third year	4	1	7
Add £50 due at the end of third year	50	0	0
Principal of fourth year	156	1	7
Interest at the end of fourth year	6	4	10
Add £50, due at the end of fourth year	50	0	0
Principal of fifth year	212	6	5
Interest at the end of fifth year	8	9	10
Add £50, due at the end of fifth year	50	0	0
Amount due at the end of five years	270	16	3

If Simple Interest alone were allowed, I should write the interest for the successive years by itself, and add its amount to the final amount;—by this means no interest would be allowed upon interest, *i. e.* there would be no Compound Interest.

The subjoined Examples are worth notice.

Ex. VII. What must I give for a freehold, let for £225 a year, so as to have $4\frac{1}{2}$ per cent. for my money? Or in other words,

If every £100 laid out bring £$4\frac{1}{2}$, what sum will produce £225? The statement will be

$$£4\tfrac{1}{2} \;:\; £225 \;::\; £100$$

and the fourth term $= \dfrac{£225}{£4\frac{1}{2}} \times £100 = \overset{25}{\cancel{225}} \times 100 \times \dfrac{2}{\cancel{9}}\,£ = £5000.$

148. Sometimes in speaking of the price of a piece of property, it is said that a certain number of years' purchase is given for it: this is the same as so many years' rental. Thus, if a field, the rent of which is £4, be sold for £100, we say that 25 years' purchase was given for it, because the price is 25 times the rental.

Ex. VIII. How many years' purchase should be paid for freehold property to clear $4\frac{1}{2}$ per cent.?

I must here see how many times a rent of £$4\frac{1}{2}$ must be repeated to produce £100, the price of the land which gives £$4\frac{1}{2}$.

This number $= \dfrac{£100}{£4\frac{1}{2}} = 100 \times \dfrac{2}{9} = \dfrac{200}{9} = 22\frac{2}{9}$; the price paid is therefore said to be $22\frac{2}{9}$ years' purchase.

Exs. 51.

1. What will an annuity of £60, payable yearly, amount to in 6 yrs. at 5 per cent. Compound Interest?

7. Find the amount due from a pension of £100, payable half-yearly, which has been unpaid for $3\frac{1}{2}$ yrs., allowing 5 per cent. Comp. Interest?

3. What principal lent for $2\frac{1}{2}$ yrs. at 5 per cent. Compound Interest will amount to £700 12*s.* $9\frac{3}{10}$*d.*?

4. What should be the purchase money of an estate, of which the rental is £5200, so that the buyer may receive $3\frac{1}{2}$ per cent. for his money?

5. A purchaser invests £7500 in land, and receives $2\frac{3}{4}$ per cent. upon his investment; what is the rent?

6. What per centage is received upon the purchase money, when an estate whereof the rent is £367 10*s.*, is bought for £10500?

7. How many years' purchase should be paid for freehold property, to produce 3¾ per cent.?

8. A freehold is sold at 33 years' purchase; what rate of interest is received on the investment?

9. What is the value of a perpetual annuity of £120 at the rate of 4½ per cent.?

10. Property which brings 7 per cent. lets for £85 15*s.*; what was the purchase money?

DISCOUNT.

149. DISCOUNT is an allowance made by a creditor to a debtor who pays a debt before it is due. When this allowance is subtracted from the debt, the remainder, *i. e.* the sum that is paid, is called the present worth.

Discount is calculated at a certain rate per cent., and in common usage is treated just the same as Interest: we shall, however, show that this is not strictly correct, but that the person who pays the money has thereby more than the just allowance made to him.

For instance; if £50 were due to me at the end of one year, but I were willing to allow a discount of 5 per cent. for ready money, then, according to the common usage, I should throw off the interest of £50 for one year, viz. £2 10*s.*, and receive only £47 10*s.* But if I make a debtor an allowance for paying ready money, I do so upon the supposition that I can place out to interest the ready money which I receive, and together with the interest can make up the £50 at the end of the year. Now, if I put out to interest £47 10*s.* at 5 per cent., I shall obtain as interest £2 7*s.* 6*d.*, and therefore I shall in all receive £49 17*s.* 6*d.*: hence I lose 2*s.* 6*d.* by this arrangement. The real question now is—What sum put out to interest for a year at 5 per cent. will amount to £50? Or—If £100 will amount to £105, what sum will amount to £50? The statement will be

Interest for 1 year £ 5
Principal £100
Amount £105 : £50 :: £100 (F)

and the fourth term will be found to be £47 12*s.* 4$\frac{4}{7}$*d.*

Also, the interest of £47 12*s.* 4$\frac{4}{7}$*d.*, for one year is £2 7*s.* 7$\frac{3}{7}$*d.*; and this, together with the principal, = £50: and therefore I neither gain nor lose.

By observing (F), we notice that the third term, £100, is the present worth of the first term, £105; and the fourth term is the present worth of the second term, £50. Hence, in any question where the present worth of a sum is required, the third term is £100; the first term is the amount of £100 at interest for the given time; and the middle term is the sum due.

150. If the discount, and not the present worth, be required, we must place in the third term the discount of £105, viz. £5. But since the discount in the third term would generally require to be reduced to the lowest denomination expressed, and the work be thereby rendered heavy; it is therefore generally better to find the present worth, and then obtain the discount by subtracting the present worth from the bill due.

151. The most common form in which discount occurs is in the use of what are called *Bills*, which are stamped papers, bearing a written engagement to pay a sum of money at a certain future time. If such a bill be presented to a banker before it is due, *i.e.* before the time fixed for pay-

ment, and the persons who are responsible for this bill are considered able to meet it at the proper time, the banker will give ready money for it, retaining, however, the *discount* upon the sum, as his remuneration for the accommodation.

In practice, as was said, it is usual to charge interest, and not discount: therefore the banker gains by the transaction, and the amount of this gain will be found to be the interest upon the true discount. For if we refer to the Ex. in (149) we shall see that in discounting a bill of £50 due in twelve months, the banker would deduct £2 10*s.*, *i. e.* the interest on £50; whereas he ought to have deducted only £2 7*s.* 7$\frac{3}{7}$*d.*; which is the interest upon £47 12*s.* 4$\frac{4}{7}$*d.*; therefore the extra sum which he takes is the interest upon £2 7*s.* 7$\frac{3}{7}$*d.*, or the interest upon the true discount.

Though discount is not in practice correctly used, yet a pupil in working Exs. should always employ the true method.

The bills mentioned above are said to be *drawn* upon the person or persons who agree to pay the money, and those who allow any such bill to be drawn on them are said to *accept* it: hence they are called acceptors, and the bill itself is called an acceptance. These acceptances are generally for any number of calendar months: but in this country three days, called *Days of Grace*, are allowed after the bill is nominally due, before it is legally due; so that a bill drawn on March 30th, at three months, would not be legally due till July 3rd.*

Ex. II. What does a banker gain by discounting a bill of £403 4*s.*, drawn Oct. 13, at four months, and discounted, Dec. 5, at 4 per cent.?

Here the bill is legally due on Feb. 16; and from Dec. 5 to Feb. 16 are 73 days:

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* A form of acceptance is given under the Art. *Book-keeping.*

	£	s.	d.
The interest for that time... =	3	4	$6\frac{18}{125}$
And the true discount ...... =	3	4	0
Therefore the banker's gain =			$6\frac{18}{125}d.$

Exs. 52.

Find the Discount on

	£	s.	d.		months		£	
1.	100	0	0	for	6	at	5	per ct.
2.	128	18	6	,,	3	,,	4	,,
3.	157	10	0	,,	9	,,	6	,,
4.	1128	17	6	,,	7	,,	$3\frac{1}{2}$	,,

Find the present worth of

	£	s.	d.		months		£	
5.	1000	0	0	due in	12	at	5	per ct.
6.	875	10	0	,,	8	,,	$4\frac{1}{2}$	,,
7.	119	6	0	,,	5	,,	$3\frac{1}{2}$	,,
8.	425	15	6	,,	$2\frac{1}{2}$y.	,,	3	,,

9. Find the difference between the interest and discount of £525 for $1\frac{1}{2}$ yrs. at 5 per cent.

What would a banker gain by discounting the following bills?—

		Drawn		Discounted	
10.	£325 8*s*. 4*d*.	March 15 at 4 months,		April 6th, at 4 per cent.	
11.	£90 7*s*. 6*d*.	Sept^r 1 ,, 9 ,,		Jan. 15th, ,, 5 ,,	

12. What is the present worth of £500, one-half of which is due in 4 months, and the remainder in 6 months, discount at $5\frac{1}{2}$ p. c. per ann.?

152. A very important application of Discount occurs in those branches of business, where it is the custom to take off 20, 30, &c., per cent. discount from the gross or invoice price of goods. And great errors may sometimes be made by tradesmen who do not know how much to add to the net value of an article, in order that they may, without loss, make the deduction agreed upon.

For instance, suppose a tradesman has an article of which the net price should be 50*s*., and it is usual to allow 20 per cent. discount, or deduct one-fifth from the invoice price. If he, thinking to allow for this discount, puts on $\frac{1}{5}$th of the 50*s*., and thus makes a gross price of 60*s*.; then, when he takes off 20 per cent. or $\frac{1}{5}$th, he will find his net price to be 48*s*., thereby losing 2*s*. But if, instead of putting on $\frac{1}{5}$th he had put on $\frac{1}{4}$th, he could then take off the required $\frac{1}{5}$th and be no loser.

			s.	d.
Thus,	Net price required.............	=	50	0
	Add $\frac{1}{4}$th of 50s.................	=	12	6
	Gross price..........	=	62	6
	Subtract $\frac{1}{5}$th of 62s. 6d.	=	12	6
	Net price as before	=	50	0

The general rule will be found as follows: Let the discount agreed upon be represented as a fractional part of £100: Thus, for 5, 10, 20, 30, 35, &c. per cent., the fractions would be

$$\frac{5}{100},\ \frac{10}{100},\ \frac{20}{100},\ \frac{30}{100},\ \frac{35}{100},\ \&c.$$

or, in their lowest terms,

$$\frac{1}{20},\ \frac{1}{10},\ \frac{1}{5},\ \frac{3}{10},\ \frac{7}{20},\ \&c. \qquad (G)$$

These are the fractional parts of the gross price to be *deducted* from it: and the corresponding frac^l parts that must be first *added* to the net price will be

$$\frac{1}{20-1},\ \frac{1}{10-1},\ \frac{1}{5-1},\ \frac{3}{10-3},\ \frac{7}{20-7},\ \&c.$$

$$\text{or}\quad \frac{1}{19},\quad \frac{1}{9},\quad \frac{1}{4},\quad \frac{3}{7},\quad \frac{7}{13}, \qquad (H)$$

where it is to be observed that the fractions in (H) have numerators the same as in (G); but the den^{rs} are the former den^{rs}, minus the respective numerators*.

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* The following demonstration of the above rule may be read by those acquainted with the elements of Algebra.

Let G = gross price; N = net price; also, let x represent the frac^l part of N to be added thereto to reproduce G, so that $N + xN = G$; or $N(1+x) = G$. (1)

Let $\frac{m}{n}$ represent the frac^l part of G to be taken off as discount so as to leave N; so that $G - \frac{m}{n}G = N$; or $G\left(1-\frac{m}{n}\right) = N$; substituting for N in (1), we have

$$G\left(1-\frac{m}{n}\right)(1+x) = G;\ \text{or } 1+x = \frac{1}{1-\frac{m}{n}} = \frac{n}{n-m};$$

$$\therefore\ x = \frac{n}{n-m} - 1 = \frac{n-n+m}{n-m} = \frac{m}{n-m};$$

i. e. if we wish to take off as discount $\frac{m}{n}$, we must put on $\frac{m}{n-m}$.

For example, if 30 per cent. discount is to be allowed upon an article of which the net price should be 21*s.*, we refer to the rows (G) and (H), and learn that if 30 per cent. or $\frac{3}{10}$ is to be taken off, $\frac{3}{7}$ must be put on: and we have as follows.

		s.
Net price	=	21
Add $\frac{3}{7}$ths of 21*s.*	=	9
Gross price charged in invoice	=	30
Subtract $\frac{3}{10}$ths of 30*s.*	=	9
Net price as before	=	21

PROFIT AND LOSS.

153. **Profit and Loss** is another application of Proportion. It is calculated at so much per cent., or at a certain *per centage*, and the general object of all Exs. under this head is, to find—(1) What per centage of profit or loss will result from selling an article at a certain price:—(2) At what price it must be sold, that there may arise a certain per centage of profit and loss; the prime cost of the article being in both cases known.

154. It will not be attempted to exhibit an Ex. of every kind of question that may arise; but a sufficient number will be given to show the principles upon which all the questions depend; and the particular method of applying the principles of Proportion in each case must be left to the judgment of the pupil.

Ex. I. If an article cost £2 7*s.* 3*d.*, and be sold for £3 3*s.* 0*d.*, what is the gain per cent.?—or, If £2 7*s.* 3*d.* become £3 3*s.* 0*d.*, what will £100 become? The statement is

£2 7*s.* 3*d.* : £100 :: £3 3*s.*

and the fourth term is £133 6*s.* 8*d.* Hence the gain upon £100 is £33 6*s.* 8*d.*; or the profit is at the rate of 33⅓ per cent.

Ex. II. If, by selling tea at 6*s.* 4*d.* per lb., a grocer lose 6 per cent., what was the prime cost per lb.?

Now, to lose 6 per cent. is to obtain only £94 for every £100 laid out; hence the question is really this—If £100 be laid out, and £94 be received for it, what is laid out when 6*s.* 4*d.* is received?

Here, since £100 is prime cost, or buying price, and we want buying price in the fourth term, we have this statement:

£94 : 6*s.* 4*d.* :: £100

and the fourth term, or prime cost of 1 lb. is 6*s.* 8$\frac{40}{47}$*d.*

Ex. III. At what price must I sell a commodity purchased at the rate of £14 5*s.* per cwt. so as to gain 21 per cent.?

In this Ex. it is required to receive £121 for £100 laid out: therefore £121 is the selling price of that which cost £100; and since the fourth term is to be the selling price of 1 cwt., we have

£100 : £14 5*s.* :: £121

and the required price is £17 4*s.* 10$\frac{1}{5}$*d.*

Questions of this kind, wherein we require the price at a certain profit per cent., may often be worked more briefly by the method of Practice. Thus to gain 20 per cent on any sum of money invested is merely to add one-fifth of the sum to the previous amount; and Ex. III. may be worked as follows.

	£	*s.*	*d.*
Prime cost	14	5	0
Add 20 per cent. or $\frac{1}{5}$th	2	17	0
Add 1 ,, or $\frac{1}{20}$th of 20 p. c.		2	10$\frac{1}{5}$
∴ original sum + 21 per cent. profit =	£17	4	10$\frac{1}{5}$

The following Ex. involves the principles of Exs. II. and III.

Ex. IV. A person, by disposing of goods for £182, loses at the rate of 9 per cent.; what should have been the selling price, so as to make a profit of 7 per cent.?

We may work this question by two operations: first, find the prime cost, and then from it find that selling price which would give a profit of 7 per cent. Since to sell at 9 per cent. loss is to receive but £91 for that which cost £100, we have this statement:

£91 : £182 :: £100

and the answer is the prime cost £200. Also, to obtain a profit of 7 per cent. is to receive £107 for that which cost £100; therefore, to find the selling price of that which cost £200, we have

$$£100 \ : \ £200 \ :: \ £107$$

and the answer is £214, the price at which the goods should be sold to make 7 per cent. profit.

In order to work the question by one statement, we may put it under this form—If goods sold for £182 bring £91 for every £100 laid out, what ought they to be sold for, so as to bring £107 for every £100?

Here we have the £100 laid out the same in both circumstances, and it will therefore not affect the question: £107 and £91 may be considered as the gaining and losing rates; also, £182 is the sum received for goods, and therefore of the same nature with the fourth term; hence the statement is

$$£91 \ : \ £107 \ :: \ £182$$

and the fourth term, or selling price, is £214, the same that was obtained from the former statements.

Questions under this head may be worked according to the principles described in the latter parts of (133) and (140).

Thus, to take a simple case, If a man buys for £4 10*s.*, and sells for £5 15*s.*, how much will he gain per cent.?

The actual gain is £5 15*s.*—£4 10*s.* = £1 5*s.*

$$\therefore \ £1\tfrac{1}{4} = \text{gain on } £4\tfrac{1}{2} = 4\tfrac{1}{2} \times \text{gain on } £1$$

$$\text{and } £\,\frac{1\tfrac{1}{4}}{4\tfrac{1}{2}} = \text{gain on } £1.$$

Let A = rate per cent. or gain on £100,

$$\text{then } £\,\frac{A}{100} = \text{gain on } £1\,;$$

$$\therefore \frac{A}{100} = \frac{1\tfrac{1}{4}}{4\tfrac{1}{2}}, \text{ or } A = \frac{100 \times 1\tfrac{1}{4}}{4\tfrac{1}{2}} = \frac{250}{9} = 27\tfrac{7}{9}.$$

Exs. 53.

1. Paid £137 12*s.* 6*d.* for goods, and sold them for £151 7*s.* 9*d.*; what was the profit per cent.?

2. By selling goods at 3*s.* 6*d.* I gain 12 per cent.; what shall I gain or lose by selling them at 4*s.* 9*d.*?

3. I give 3*s.* 9*d.* for goods; at what rate must they be sold to make a profit of 30 per cent.?

4. Bought cloth at 9*s.* 4½*d.* per English ell; it is required to find the selling price per yard so as to gain 17½ per cent.

5. Goods were bought for 2*s.* 9*d.*, being 17½ per cent. below their real value; what was that value?

6. Sold goods for £3 13*s.* 6*d.*, being 22½ per cent. profit; what was the prime cost?

7. I sell an article for £22 10*s.*, and by so doing lose 15 per cent.; what per centage would be lost or gained by selling it at £27?

8. At a selling price of 15*s.* I lose 10 per cent.; what must be the price to gain 10 per cent.?

9. I buy tobacco at 10 guineas per cwt.; at what price must I retail it per lb. so as to gain 12 per cent.?

10. When the price of a certain article is 12*s.* 6*d.* there is a gain of 25 per cent.; what would be the loss or gain if the price were 10*s.*?

11. I bought 145 quarters of wheat at 50*s.* per quarter, and in selling I make a profit of £36 5*s.*; how much per cent. was the profit?

12. A merchant sold a pipe of wine for £50, and by so doing lost 5 per cent.; at what price must he sell 3 other pipes so that he may gain 5 per cent. upon the prime cost of the 4 pipes?

13. A person having bought goods for £20, sells half of them so as to gain 10 per cent.; for how much must he sell the remainder so as to gain 20 per cent. upon the whole

14. I bought 56 gallons of brandy at 22*s.* 6*d.* per gallon, but 7 gallons were lost; at what price per gallon must I sell the remainder, to obtain 15 per cent. profit on the whole outlay?

PARTNERSHIP.

155. PARTNERSHIP, or, as it is sometimes called, FELLOWSHIP, is the Rule by which we determine how to divide profits, which arise from different sums of money put into a business by two or more persons, either for the same or different periods of time.

Ex. I. Two persons enter into business as partners; one puts in £350, and the other £500; they gain £100. How is the profit to be divided?

Here the profit, £100, is made from the whole capital, £850; and each partner's share of the profit will be in proportion to his share of the capital: therefore the question divides itself into these two parts:—(1) If £850 produce a profit of £100, how much will £350 produce? (2) If £850 produce £100, what will £500 produce?

The statements for these two questions will plainly be

$$\begin{array}{lccccc} & \pounds & & \pounds & & \pounds \\ (1) & 850 & : & 350 & :: & 100 \\ (2) & 850 & : & 500 & :: & 100 \end{array}$$

$$\begin{array}{rrrr} & \pounds & s. & d. \\ \text{and the fourth term of (1)} = & 41 & 3 & 6\frac{6}{17} \\ \text{,, \quad ,, \quad ,, (2)} = & 58 & 16 & 5\frac{11}{17} \\ \hline \text{and these together} \ldots\ldots = & 100 & 0 & 0 \end{array}$$

Ex. II. A field of grass is rented by two persons for £27; the one keeps in it 15 oxen for ten days, and the other 21 oxen for seven days: find the rent to be paid by each, supposing the pasturage to remain equally good throughout?

Here it is plain that the keep of 15 oxen for ten days is the same as of ten times 15, or 150, oxen for one day: so also, of 21 oxen for seven days is the same as of 7 times 21, or 147, for one day; therefore the question is plainly this: If one man turn into a field 150 oxen, and another 147, for one day, and they together pay £27, how is that payment to be divided?

The whole number turned in would be 297, and the two statements would be similar to those in Ex. I., viz.

$$\begin{array}{cccccc} 297 & : & 150 & :: & \pounds 27 & (1) \\ 297 & : & 147 & :: & \pounds 27 & (2) \end{array}$$

the fourth term of $(1) = \frac{150 \times \cancel{27}}{\underset{11}{\cancel{297}}} \pounds = \pounds 13 \ 12s. \ 8\frac{8}{11}d.$

,, ,, ,, $(2) = \frac{147 \times \cancel{27}}{\underset{11}{\cancel{297}}} = \pounds 13 \ 7s. \ 3\frac{3}{11}d.$

and their sum .... $= \underline{\underline{\pounds 27 \ \ 0 \ \ 0}}$

156. I will give one more Ex. which is the same in principle as Exs. I. and II., but is more complicated in its operations.

Ex. III. On the 1st of January *A* brought into a business £350, and on the 1st of April £500 more: on the 1st of June he takes out

£400; three months after this he brought in £600. *B* brought into the business £500: four months after this he takes out £150; and on the 1st of November he brought in £650. At the end of the year their clear gain is £1008. How much ought each to receive?

Here *A* put in £350 from January 1st to June 1st, or five months; also, £500 from April first to June 1st, or two months: he has now in the business £850, but he takes out £400, leaving £450. This £450 is in from June 1st to December 31st, or seven months. Also, he has £600 in from September 1st to December 31st, or four months. Hence he has in all

£			£		
350	for 5	months, or	1750	for 1	month
500	,, 2	,, or	1000	,, 1	,,
450	,, 7	,, or	3150	,, 1	,,
600	,, 4	,, or	2400	,, 1	,,
Therefore he has in all			8300	,, 1	,,

Again, *B* brought in £500 from January 1st to May 1st, or four months: he now takes away £150, and has in £350 from May 1st to December 31st, or eight months. Also, he brings in £650 from November 1st to December 31st, or two months. Hence he has

£			£		
500	for 4	months, or	2000	for 1	month
350	,, 8	,, or	2800	,, 1	,,
650	,, 2	,, or	1300	,, 1	,,
Therefore *B*'s capital is			6100	,, 1	,,
and *A*'s capital was			8300	,, 1	,,
therefore the joint capital =			14400	,, 1	,,

Hence the two statements will be

	£		£		£
For *A*'s share	14400	:	8300	::	1008
For *B*'s ,,	14400	:	6100	::	1008

and the fourth terms are £581 for *A*, and £427 for *B*.

157. In this place we may introduce an Ex. of the following kind.

Ex. IV. A wine merchant mixes together 20 gallons of wine at 12*s.* a gallon, 25 gallons at 14*s.*, and 36 gallons at 16*s.*: what should be the price of a gallon of the mixture?

Here it is plain that	20	gallons at	12*s*.	are worth	240*s*.
also	,, 25	,,	14*s*.	,,	350*s*.
and	,, 36	,,	16*s*.	,,	576*s*.
and therefore that	81	of the mixture		,,	1166*s*.

hence the value of one gallon is plainly $\frac{1}{81}$th of the value of the whole;

or, price per gallon $=\frac{1166}{81}$ *s*. $=14$*s*. $4\frac{20}{27}$*d*.

158. The following Ex. shows how to divide a given quantity into parts which shall have to each other given ratios. It is upon the same principle as the previous Exs., though not commonly recognized as such.

Ex. V. Divide 1065 into parts which shall be to each other in the ratio of 3, 5, 7; and also into parts which shall be in the ratio of $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$.

Taking the former part of the Ex., we observe that $3+5+7=15$ is the smallest integer which can be divided in the ratio of 3, 5, 7; hence the required portions of 1065 will bear the same ratio to 1065 that 3, 5, 7 do respectively to 15. The first portion is obtained by the statement

$$15 \ : \ 3 \ :: \ 1065;$$

and the fourth term $=213$. Also, if 5 and 7 be successively placed in the second term of the above statement, we shall find the remaining portions to be 355, and 497.

Similarly, in the latter part of the Ex., we see that $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$, or $\frac{35+21+15}{105}=\frac{71}{105}$ is a fraction which can be broken up into the fractions $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$; hence the required portions of 1065 will bear the same ratio to the entire number 1065 that $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$, respectively bear to their sum $\frac{71}{105}$; we have, therefore, as the statement for obtaining the first portion

$$\frac{71}{105} \ : \ \frac{1}{3} \ :: \ 1065;$$

and the fourth term $=\frac{\overset{\overset{6}{\cancel{15}}}{\cancel{1065}}}{1}\times\frac{1}{\cancel{3}}\times\frac{105}{\cancel{71}}=525$. So also the second and third statements will be

$$\frac{71}{105} : \frac{1}{5} :: 1065;$$

$$\text{and } \frac{71}{105} : \frac{1}{7} :: 1065;$$

and the corresponding fourth terms will be found to be 315 and 225.

Exs. 54.

1. Two persons invest in business £300 and £250 respectively; they gain £150: how is it to be divided?

2. *A* and *B* as partners lost £600; if *A*'s capital were £4000, and *B*'s £2100, how much of the loss must each bear?

3. *A*, *B*, and *C* were partners: *A* put in £1000 for 2 yrs., *B* £750 for 15 months, and *C* £1500 for 9 months; divide equitably a profit of £1000.

4. *A*, *B*, and *C* rent a field for £10; *A* put in 20 horses, *B* 15 oxen, and *C* 10 sheep; how should the expence be divided, if the eating of a horse, ox, and sheep be in the ratio of 3, 2, and 1?

5. *A* puts into a concern £500, and 6 months after puts in £300 more; *B* puts in £1000, and 3 months after puts in £1000 more; they trade for 2 yrs., and gain £650: what is the share of each?

6. Divide £20 amongst 3 persons, so that their shares shall be in the ratio of 3, 4, and 5.

7. Distribute £705 in the ratio of $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$.

8. A mixture is made of 5lbs. of tea at 3*s*. 4*d*., 10lbs. at 4*s*. 2*d*., and 15lbs. at 5*s*.: what should be the price per lb. of the mixture?

9. A wine merchant mixes together 20 gallons at 12*s*., 30 gallons at 15*s*., 40 gallons at 20*s*., and 10 gallons of water; what must be the retail price so as to gain 10 per cent. profit?

10. A testator bequeaths £1000 to one person, £500 to a second, and £300 to a third; but his property is found to realize only £1250: how much should each receive?

Exs. 55. I.

1. If 375 quarters are grown on 75 acres of land, how much land will be required to grow a peck?

2. Ten persons joined to buy three lottery tickets for £10, £25, and £40: the second gains a prize of £1000, how much does each man gain?

3. How many times does a clock tick in the month of January, if it ticks 15 times in 2 minutes?

4. A large bin contains 15 cub. ft. 267 cub. in.; and out of it 3 smaller bins are filled, each containing 4 cub. ft. and 375 cub. in.; how much will be left in the bin?

5. Out of a square plate of metal $15\frac{3}{4}$ feet long, how many circular pieces can be cut, $2\frac{1}{4}$ inches in diameter?

6. What is the smallest sum that a man must have in his pocket that he may be able to pay it away entirely either in moidores, guineas, marks, or 7 shilling pieces?

7. Assuming the method of multiplying and dividing fractions by whole numbers, shew that $\frac{3}{5} \div \frac{5}{9} = \frac{27}{25}$.

8. If 9 slabs, 12 in. long, and 12 inches broad, will cover a certain surface, how many slabs will be needed if they be 18 in. long, and 8 in. broad?

9. Find the cost of 8725lbs. at 1*s*. $10\frac{3}{4}$*d*.

10. What cost 25oz. 3dwts. 11grs. at £3 17*s*. $10\frac{1}{2}$*d*. per oz.?

11. If a tradesman gain 1*s*. 6*d*. on an article which he sells for 5*s*. 9*d*., what is the profit per cent. on the prime cost?

12. Of two pieces of cloth, one is 42 in. wide, and costs 1*s*. 6*d*. per yd.; the other is 56 in. wide, and costs 2*s*. 4*d*. per yd.; what is the ratio of the qualities of the pieces, supposing the prices to be exactly proportionate to their real value?

13. Silver coinage has 37 parts pure silver, and 3 parts of copper; 1lb. Troy makes 66*s*.; what quantity of pure silver is there in 20*s*.?

14. Reduce 8 cwt. $3\frac{1}{2}$ qrs. $1\frac{1}{5}$ stones to the decimal of 15 tons.

15. What decimal of a square furlong is 1 perch?

16. If the net price of an article be found by taking off $\frac{2}{15}$ths of the gross price, what must be added to the net price to make the invoice price?

17. If I am required to throw off $\frac{1}{15}$th, $\frac{2}{21}$ths, $\frac{3}{17}$ths, from the invoice price of three parcels of goods, what must be the fractional parts to be added to the required net prices?

K.

1. How many strokes will a clock strike in the month of May?

2. Find the value of 4424 articles at 15*s*. $10\frac{1}{2}$*d*. each.

3. A block of stone 7 ft. long, $2\frac{1}{2}$ ft. broad, and 15 in. thick, weighs $3\frac{1}{4}$ tons; what must be the length of a block of the same kind, whereof the breadth is $3\frac{1}{4}$ ft., the thickness 10 inches, and weight 6500 lbs.?

4. What is the cost of 59 tons, 15 cwt., 3 qrs., 18 lbs., at £26 18*s*. 9*d*. per ton?

5. A person buys 68 yards of cloth for £75, and retails it at £1 18*s.* per English ell; what does he gain by the transaction?

6. What may a person spend per day out of an income of £1000 a year, if he lay by 20 guineas every calendar month?

7. Explain the reason of stating a Rule of Three sum; and shew why the answer results in the same name as the third term.

8. A tax of 3*d.* in the pound on a certain assessment produces £1080; how much will be produced by a tax of 7*d.* on an assessment of double the value?

9. What is the value of $(\frac{4}{5}+\frac{3}{8}-\frac{5}{12})$ of £360?

10. Explain the nature and advantages of Decimal Fractions, compared with Vulgar Fractions.

11. What fraction of a square mile are $2\frac{1}{4}$ perches?

12. I have to distribute 150 yds. to 10 men and 10 women, so that the men and women may have shares in the ratio of 2 : 3; how much will each have?

13. A boat is propelled by 8 oars, which take 10 strokes per minute; and it goes 9 miles an hour; find the rate of a boat propelled by 6 oars which take 8 strokes per minute, when 5 of its strokes are equal to 6 of the former.

14. Find the exact value of

$$\cdot375 \text{ of } 6s.\ 8d. - \cdot941875 \text{ of } 4s. + 1\cdot9898\dot{3} \text{ of } 2s.$$

15. Simplify the following expression:

$$\frac{1}{2}+\frac{1}{3\times2^3}+\frac{1}{5\times2^5}+\frac{1}{7\times2^7}.$$

16. If I throw off 17 per cent. from my invoice price, what fractional part of my net price must I put on?

L.

1. The quotient is 3276, the divisor £2 7*s.* 6*d.*, remainder 6*d.*; find the dividend.

2. On the 1st of March I borrow £10, to be repaid in a calendar month, and in return lend £15 on the 1st of April; when should it be repaid?

3. The girth of a tree at the surface of the ground is 6 feet, find the girths at 10 feet and 20 feet, if the height of the tree be 50 feet, and it tapers regularly.

4. Find the cost of $13\frac{7}{16}$ oz. at £3 7*s.* 6*d.* per oz.

5. At what distance from the end of a slab of 17 in. breadth must I cut, so as to have a rectangular piece, containing half a square yard?

6. An author pays 2*s.* 6*d.* for the printing &c. of a book: out of the publishing price 10 per cent. is allowed for advertising, 10 per cent. for publisher's commission, 25 p. c. to the retail trade, 4 p. c. for damaged copies, and 5 p. c. loss of interest; what must be the publishing price, so that he may neither gain nor lose?

7. In the last question, what would he gain on 2000 copies, at a selling price of 5*s.* 6*d.*?

8. If 56 current shillings be worth £$2\frac{7}{18}$ in gold, how many current shillings are worth £65 in gold?

9. Two numbers are to one another as 8 : 11; and the greater one is 77; find the less.

10. Find the whole cost of a house, of which the rent is £27; the poor rate 3*s.* 4*d.* in the pound; gas rate two-thirds of the poor rate; and the paving rate three-fifths of the gas rate.

11. Reduce to a vulgar fraction

$$\frac{2{\cdot}3}{1{\cdot}\dot{7}} \times \frac{13{\cdot}\dot{8}\dot{5}}{1{\cdot}0\dot{2}} \times \frac{1{\cdot}21}{4{\cdot}9}.$$

12. Express the ratio of £$3{\cdot}\dot{7}$ to $4{\cdot}1\dot{5}$ guineas in the smallest integers.

13. A piece of work employs 15 men for 6 days, when the day is 12 hours long, and costs £$18\frac{5}{8}$; what will be the cost of a piece employing 25 men for 9 days of 10 hours each, the pay of the new workmen being $1\frac{1}{5}$ times that of the old?

14. A tank is filled by 3 pipes in 2, $8\frac{2}{7}$, and $7\frac{1}{2}$ hours respectively; in what time would they all fill it?

15. A discount of $\frac{2}{13}$ths is agreed upon; what must be the ratio between the net and gross prices, so that I may be able to make the deduction?

16. I add $\frac{7}{13}$ths to the net price of an article; what per centage of discount was agreed upon?

STOCKS.

159. The government of a country sometimes finds it necessary to borrow money; and it gives to the lender a bond acknowledging the debt, and agreeing to pay a certain rate of interest for the money. The amount owing to those who

hold these bonds is called the National Debt, or *The Funds;* and the interest paid is derived from the income of the country, arising principally from taxation.

The above-mentioned bonds are saleable, and are called *Stock;* and they of course vary in value, principally according to the plentifulness or scarcity of money.

Thus, suppose a person lend £100 to government, and receive an acknowledgment for it, with an agreement to pay £3 a year interest for the loan; then, if at any time he wishes to sell the bond, and money is scarcer than when he lent the £100, he will get less for it than he gave, perhaps £95; and if money be more plentiful, he will get more for it, perhaps £105. But still the bond represents an acknowledgment for £100, and £3 interest is paid to the holder of it, whatever may be the sum which he has paid for it. When, therefore, we say that 3 per cent. stock is selling at 80, we mean that the buyer of £100 bond, or, as it is called, £100 stock, has to give only £80 sterling for it; hence, as he gets £3 interest for it, it is not £3 *per cent.* to him, but £3 for £80. From this it is plain that the lower the stock is in price, the better interest the buyer obtains; and the higher the stock is, the less interest he obtains for his money.

When the market price of £100 stock is exactly £100, the stock is said to be at *par;* if the price be more than £100, it is said to be at a *premium;* if below £100, at a *discount.* The smallest variation in the price of stock is one-eighth of £1, or 2*s.* 6*d.*, for every £100 stock.

160. Since persons who wish to sell stock may not know any who wish to buy, therefore all sales and purchases are transacted through agents, who are called Stock-brokers. The broker of the buyer deals with the broker of the seller, and each charges his employer, or principal, as he is called, a

commission of ⅛th of £1 for the transfer of every £100 stock; so that a buyer must always consider that he pays ⅛th per cent. more, and the seller that he receives ⅛th per cent. less than the selling price; but if in any Ex. the commission be not mentioned, no notice need be taken of it.

161. In working the various questions that occur in Stocks, a pupil must be careful not to confound stock and actual money. Also, in buying or selling stock, it is quite immaterial whether it be 3 per cent., 4 per cent., or any other kind of stock, unless we wish to know the *income* to be derived: for instance, if I have to sell out £100 stock, when the price is 95, it matters not whether it be in the 3, 4, or 5 per cents.: I have to receive £95 for every £100.

Ex. 1. What must be given for £5050 stock in the Three per Cents., at 85⅜ per cent.?

Here the price of £100 stock is 85⅜; and I have to find the price of £5050 stock: hence the statement will be

£100 : £5050 :: £85⅜.

Multiplying by 85⅜, after the method of (84, Ex. IV.), and then dividing by 100, we have the annexed operation, in which we find the cost of £5050 stock at 85⅜ = £4311 8*s.* 9*d.*

```
                               5050
                                 85⅜
                              -------
                              25250
                             40400
2/8, or 1/4, of £5050 =        1262  10
1/8 of £5050          =         631   5
                       ----------------
                   1,00) 4311,43  15
                                20
                              ----
                              8,75
                                12
                              ----
                              9,00
```

Next, let us find what amount of stock any given sum will buy, when the price of £100 stock is known.

Ex. II. How much stock can be bought for £1490, the price being 88⅝, and commission ⅛ per cent.?

Adding the commission to the market price, we have the cost to the buyer $88\frac{4}{8}$, or $88\frac{1}{2}$. The term sought is amount of stock; and in the proposed question £100 is the stock to be bought by $£88\frac{1}{2}$: we have therefore

$$£88\tfrac{1}{2} \;:\; £1490 \;::\; £100$$

and the fourth term, or amount of stock bought by £1490 is £1683 12*s.* $3\frac{47}{59}$*d.*

Ex. III. If a person invest £2000 in the Three per Cents. when they are at $95\frac{1}{2}$, what is his annual income therefrom?

In this case the buyer gives $£95\frac{1}{2}$ for £100 stock, *i. e.* for the privilege of receiving £3 interest: hence the question is,—If $£95\frac{1}{2}$ produce £3 interest, what will £2000 produce? Our statement is

$$£95\tfrac{1}{2} \;:\; £2000 \;::\; £3$$

and the fourth term, or interest of £2000, will be found to be £62 16*s.* $6\frac{102}{191}$*d.*

Ex. IV. Find what per centage will be obtained by investing in the Three and a half per Cents. at 91: or, in other words,—If £91 give $£3\frac{1}{2}$, what will £100 produce? The statement is

$$£91 \;:\; £100 \;::\; £3\tfrac{1}{2}$$

and the fourth term $= \dfrac{3\frac{1}{2}\times 100}{91}\,£ = \dfrac{\cancel{7}}{\cancel{2}} \times \dfrac{\overset{50}{\cancel{100}}}{\underset{13}{\cancel{91}}}\,£ = £3\tfrac{11}{13}$.

Also, transferring the 100 from the left-hand numerator to the right-hand denominator, I have

$$\frac{3\frac{1}{2}}{91} = \frac{3\frac{11}{13}}{100}.$$

Observing this equation, I notice that the right-hand side gives the ratio of the interest of £100 to £100; and the left-hand side gives the ratio of the interest on £100 *stock* to the price of that stock. Now, in finding the per centage which any interest produces, we wish to know what is the ratio of the interest on £100 to £100; and since the right-hand fraction gives this ratio, therefore the former fraction also gives it: hence this fraction gives a standard, by which we can compare the value of the per centage derived from any two investments in different kinds of stock. Thus, if I wish to know whether it will be more advantageous to invest in the Four per Cents. at 95, or in the Three per Cents. at 85, I must compare the fractions $\frac{4}{95}$ and $\frac{3}{85}$.

By (46), the difference is $\frac{4\times 85-3\times 95}{95\times 85}=\frac{340-285}{95\times 85}$. Hence, since $340 > 285$, therefore $\frac{4}{95}>\frac{3}{85}$; or an investment in the Four per Cents. at 95 will produce better interest than in Three per Cents. at 85.

If we wish to know how much better interest is obtained in the one case than in the other, we must observe that these fractions $\frac{4}{95}$ and $\frac{3}{85}$ express the respective portions of £100 which the two investments give as interest per cent.:

hence, the difference of the per centage obtained $=\frac{340-285}{95\times 85}$ of £100

$$=\frac{\overset{11}{\cancel{55}}\times\overset{20}{\cancel{100}}}{\underset{19}{\cancel{95}}\times\underset{17}{\cancel{85}}}\text{£}=\frac{220}{323}\text{£}=13s.\ 7\tfrac{151}{323}d.$$

162. We will now give an Ex. combining two or more of the operations exhibited in the previous Exs.

Ex. V. A person transfers £1000 stock from the Four per Cents. at 90 to the Three per Cents. at 72; find how much of the latter stock he will hold, and the alteration made in his annual income.

The first part of the question may be thus expressed: "If a certain sum of money will buy £1000 stock at 90, how much can be bought when the stock is at 72?" The statement will be

$$£72 \ : \ £90 \ :: \ £1000$$

and the fourth term $=\frac{\overset{125}{\cancel{1000}}\times\overset{10}{\cancel{90}}}{\underset{\cancel{8}}{\cancel{72}}}\text{£}=1250\text{£}.$

To find the income derived from the £1000 stock and from the £1250 stock, two simple statements might be employed: but where, as in this case, the stock consists of £100 shares, we can work more briefly thus:—£1000 stock = 10 cents.; and since each cent. produces £4, the whole 10 produce £40. Also, £1250 stock = $12\frac{1}{2}$ cents.; and since each cent. of this stock produces £3, the whole produce $12\frac{1}{2}\times 3\text{£}=\frac{25}{2}\times 3\text{£}=\frac{75}{2}\text{£}$ = £37 10s.; hence the difference of the incomes from the two investments = £40 − £37 10s. = £2 10s.

Exs. 56.

1. What must be given for £2000 Stock when the funds are at 85?

2. When $3\frac{1}{2}$ per cent. Stock is at $93\frac{1}{2}$, what sum will buy £1250, allowing $\frac{1}{8}$ per cent. for brokerage?

3. How much Stock can be bought for £1176 10*s.* when the funds are at £$90\frac{3}{8}$, and broker's charge 2*s.* 6*d.* per cent.?

4. What sum must be invested in 3 per cent. Stock at $94\frac{1}{2}$, to yield an annual income of £500?

5. If a person invest in the 3 per cents. at 93, at what rate per cent. will he receive interest for his money?

6. A person lays out £1000 in $3\frac{1}{2}$ per cent. Stock, when the funds are at $92\frac{3}{8}$; what income does he derive from it?

7. A sum of £999 19*s.* $11\frac{1}{7}$*d.* in the $3\frac{1}{2}$ per cents. produces £44 0*s.* 6*d.*; what was the price of the Stock when the money was invested?

8. In what Stock must I purchase, so that I may derive an income of £75 from the investment of £1875 at par?

9. A capitalist invests for a short period £100,000 in 3 per cents. at $87\frac{1}{2}$: when he sells out, they have risen 2 per cent.; what does he gain, reckoning $\frac{1}{8}$ per cent. for brokerage, both in buying and selling?

10. What must be the price of a railway share, paying a 5 per cent. dividend, and of which the nominal value is £100, so that a purchaser may receive 7 per cent. for the money invested?

11. A railway share, originally costing £100, has paid a dividend of 8 per cent.: what must I give for such a share, so as to receive $4\frac{1}{2}$ per cent. for my money?

12. Railway shares which were purchased at a discount of $10\frac{3}{4}$ per cent., and sold at a premium of £$31\frac{1}{8}$, realised a profit of £357 2*s.* $10\frac{2}{7}$*d.*: how much was invested?

13. A person having an annual income which arises from £450 Stock invested in the 3 per cents., exchanges it for an annual income arising from £315 Stock in the 4 per cents.: what is his annual gain or loss by the exchange?

14. What is the interest for $5\frac{3}{4}$ yrs. of £293 invested in the 3 per cents., when they were at $87\frac{7}{8}$?

15. A person sells out of the $3\frac{1}{2}$ per cents. at $93\frac{1}{2}$, and realizes £18700: if he invest one-fifth of the produce in the 4 per cents. at 96, and the remainder in the 3 per cents at 90, find the alteration in his income.

16. Which is the better investment, to buy in the 3 per cents. at 85, or the 4 per cents. at 102; and by how much?

17. Find the difference of income arising from two investments of £5000: (1) in shares at 131¼, paying a 6 per cent. dividend; (2) in Bank Stock at 194½, paying an 8 per cent. dividend.

18. A person wishes to bequeath an annuity of £100 a year; what sum must he devote to the purchase of 3¼ per cent. Stock at 97, so that the annuitant may receive the £100 free of 3 per cent. income tax?

EQUATION OF PAYMENTS.

163. If a person owe another several sums to be paid at different times, and it is required to know at what time it would be just to pay the whole at one payment, this would be a question to be solved by a Rule called Equation of Payments.

To ascertain the method of finding this time of payment, called the equated time, let us take a simple Example.

Ex. I. If £100 be due at the end of six months, and £200 at the end of twelve months; find when it is just to pay the whole in one sum.

It is quite clear that the time of payment will be at some period between the two fixed times, six months and twelve months; hence the former sum, £100, will be paid after it is due, and the latter sum, £200, before it is due.

Now, a person keeping the £100 beyond the appointed time ought, of course, if that were the only money to be paid, to pay interest for it; but, instead of paying interest, he is to make up for the privilege for keeping the £100 by paying the £200 before it is due; it is hence quite clear that he must pay this £200 such a time before it is due, that the interest of the £200 for that time shall just balance the interest he might obtain by keeping the £100 after it was due. The question then really is—How soon will the interest upon £200 produce the same as the interest upon £100? The answer evidently is, in half the time; *i.e.* the time of paying the £200 must be *earlier* than its original time of twelve months, by half as much as the time of paying the £100 is *later* than its original time of six months; therefore, if the payer keep the £100 *four* months beyond

the six months, and pay the £200 *two* months sooner than the twelve months, the interest gained in the one case and lost in the other will be just balanced; and the whole sum will have to be paid in 10 months.

164. Putting the question in another form, we may consider that when the sums were to be paid at different times, the payer had the £100 in his hands six months, *i. e.* he had the interest of £100 for six months, or of £600 for one month: also, he had the interest of £200 for twelve months, or of £2400 for one month; therefore he had in all the interest of £2400 + £600, or of £3000, for one month. If, then, the debtor has to pay the £300 in one sum, how long ought he to keep it, so that its interest shall equal the interest of £3000 for one month? The answer to this question will be obtained from the following statement:

£300 : £3000 :: 1 month;

and the fourth term will be ten months, the same result as before.

165. The truth of the above method has been disputed by arithmeticians upon this ground. Referring to the above Ex., it is said, that since the £100 is paid after it is due, the payer should pay the interest for the time that it is kept back; but since the £200 is paid before it is due, *discount* only should be allowed thereon: and since the discount is less than the interest, it is said that the payer, by the above method, receives as much more than his due as the interest of £200 exceeds the discount; hence, to make the payment perfectly correct according to this view, we ought to place the payment earlier than ten months. But since we have shown, by the working of the Ex. given above, that the interest gained in one case and lost in the other is equal, we shall, by placing the time for payment of the £300 earlier, rob the payer in two ways—1st, by depriving him of the

interest of the £100 during the latter part of the four months that he ought to be allowed to hold it: and 2ndly, by making him put the £200 into the hands of his creditor for a longer time than two months.

The fallacy of the reasoning which would place discount instead of interest in the question may be shown thus. It is here assumed that discount, not interest, is applicable to all cases where money is to be paid before it is due. Now, this is quite true where only one payment is to be made; for in that case we have laid it down, that the debtor is to pay such a sum as put out to interest shall just amount to the sum due: therefore, in all cases of discount the debtor pays less than he owes. But this is not the case in Equation of Payments: for here the creditor receives the whole of the latter portion of the debt, say £200, *though it be before it is due*, and he can put the *whole* out to interest, which he cannot, in real questions of discount; and the question now is,—not how much must be put out to interest to raise £200,—but how long must the whole £200 be put out, to raise an amount of interest equal to that which the creditor has lost, by allowing the first payment, say £100, to remain in the debtor's hands beyond its time. The question considered in this view has been satisfactorily answered in the Ex. worked above.

It should however be added that, strictly speaking, simple interest is almost a fiction; for the moment any sum of money is payable as interest, it matters not whether it be called principal or interest, it is the property of the lender; and if this be retained by the borrower, it ought, in justice to the lender, to be added to the principal, and be charged with interest afterwards*.

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* See Art. *Rebate* in the "Penny Cyclopædia."

Ex. II. A person owes £800; £200 to be paid in three months, £100 in four months, £300 in 5 months, and £200 in six months: if the whole were to be paid at once, what would the time of payment be?

Here, by the agreement, the debtor has the interest of

	£			£		
	200	for 3 months,	or	600	for 1 month
also of	100	,, 4 ,,	or	400	,, 1 ,,
also of	300	,, 5 ,,	or	1500	,, 1 ,,
lastly, of	200	,, 6 ,,	or	1200	,, 1 ,,
	800			3700	

therefore he has altogether the interest of £3700 for one month; and the whole sum to be paid is £800: hence, we have to find in how many months £800 will produce as much interest as £3700 in one month. The result is obtained from the following statement:

$$£800 : £3700 :: 1 \text{ month};$$

and the fourth term $=\frac{3700}{800}\times 1$ month $=4\frac{5}{8}$ months.

In the statements which are used in Exs. I. and II. we observe that the third term is one month; hence, since this will in every Ex. be the third term, we shall merely have to divide the second term by the first, in order to obtain the fourth term: also, the first term is the whole amount due; and the second is the amount of all the several sums, when each one is multiplied by its time of payment. Hence the equated time is found by the following Rule:

Multiply each sum into its specified time, and divide the total of the products so found by the whole amount to be paid.

Exs. 57.

1. Find the equated time of payment of £160, whereof £100 is due in 3 months, and £60 in 8 months.

2. I owe £100 in 2 yrs., £230 in $2\frac{1}{2}$ yrs., and £280 in 3 yrs. hence: when must I pay the whole in one sum, so as neither to gain or lose?

3. Of a debt, $\frac{1}{3}$ is due in 4 months, $\frac{1}{4}$ in 6 months, $\frac{1}{5}$ in 8 months, and the remainder in 12 months; find the equated time of payment.

4. Find the equated time of paying £200, which is due in monthly instalments of £20.

EXCHANGE.

166. **Exchange**, in its simplest meaning, is merely the conversion of any sum of money from the coinage of one country to that of another. To perform this conversion, we must of course know for how much a certain coin of one country can be exchanged in coins of another: for instance, if I wish to exchange a sum of English money for French, I must know how many of the current coins of France, viz. *francs*, will be given me for £1 of English money. This rate of Exchange between two countries is called the *Course of Exchange*, but it is not always the same; and we shall presently show the cause of the variation.

Ex. I. Exchange £850 sterling for francs at 25 francs 15 centimes, or cents; or, in other words,—If £1 sterling can be exchanged for 25 francs 15 cents, what number of francs can be obtained for £850? To obtain this number I have the following statement:

£1 : £850 :: 25 francs 15 cents.

Also, since 100 cents make one franc, therefore cents may be expressed as decimal parts of a franc; *i. e.* 25 francs 15 cents = 25·15 francs;

$$\text{hence, the fourth term} = \frac{£850}{£1} \times 25{\cdot}15 \text{ francs}$$
$$= 21377{\cdot}5 \text{ francs}$$

or, 21377 francs 50 centimes.

Ex. II. How many pounds sterling can be obtained for 8457 marks $15\frac{1}{2}$ schillings, Hamburgh, at the rate of 13 marks 12 schillings for £1 sterling? The statement is

13 marks 12 sch. : 8457 mks. $15\frac{1}{2}$ sch. :: £1.

By Tables of Hamburgh Coinage, we find that 16 schillings = 1 mark: hence, reducing the first and second terms of the above statement to half schillings, we obtain by the usual process the fourth term = £615 2*s*. 6*d*.

167. The method of these two Exs. will enable us to convert any sums from the currency of one country to that of another, when the course of exchange between the two countries is known. But we must also be able to perform this conversion between two foreign countries.

Ex. III. Change 1932 florins at Amsterdam for ducats at Naples, the course of Exchange being $80\frac{1}{2}$ florins for 40 ducats. The statement is

$$80\tfrac{1}{2}\text{ florins} \;:\; 1932\text{ florins} \;::\; 40\text{ ducats};$$

and the fourth term will be found to be 960 ducats.

In the following Ex. the process is not so simple:

Ex. IV. Exchange 1000 American dollars for sterling money when Eng. money bears a premium of 10 p. c. in America.

Here, £110 worth of dollars in America would produce only £100 of English money—hence 1000 dollars must be reduced in the ratio of 110 to 100. Thus,

$$110 \;:\; 100 \;::\; \overset{\text{dols.}}{1000} \;:\; \frac{10000}{11}\text{ dols.}$$

$$\text{or } 909\tfrac{1}{11}\text{ dols.}$$

We have now to convert $909\frac{1}{11}$ dols., each 4*s.* 6*d.*, into pounds sterling.

The required number of pounds sterling

$$= \frac{10000}{11} \times \frac{4\frac{1}{2}\text{sh.}}{20\text{sh.}} = \frac{9000}{44} = 204\tfrac{6}{11}$$

$$\text{or } £204\ 10s.\ 10\tfrac{10}{11}d.$$

If I had wished to change Eng. money for American, I should have *increased* the number of English pounds in the ratio of 100 to 110.

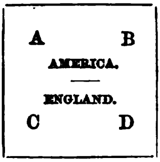
We now proceed to explain a further and more comprehensive meaning of the term Exchange.

In commercial transactions between different countries it is not usual to pay for goods imported, in coin, or as it is sometimes called, in *specie* or *bullion:* and for two reasons—first, the quantity of foreign goods imported by a country like England is so great, that if paid for in coin, the payment would speedily drain all the coin out of the country, and business could not be carried on. Secondly, there would be the probable loss of the coin by wreck or otherwise in the transmission; besides that there would arise a loss of interest on the money while it was being sent to its destination.

We shall now show how these difficulties may be avoided, when we are dealing with countries which send goods as well as receive them, *i. e.* which export as well as import.

168. The following is a simple Ex. of the manner of conducting these transactions.

Suppose *A* and *B* to represent two merchants in America, and *C* and *D* two others in England: let *C* buy of *A* a thousand pounds worth of goods, and therefore owe him £1000; so, also, let *B* owe *D* £1000 by a similar purchase; then if these sums be paid in coin, £1000 must cross the Atlantic twice. But, since *C* has to pay £1000 to *A*, he would as readily pay it to *D* in England, if by such payment he could get rid of his liability to *A*: so also *B* would pay *A*, if he could be rid of his debt to *D*. This simple transaction might, therefore, be completed thus:



Let *C* send to *A* a bill acknowledging the debt of £1000, and promising to pay £1000 to any one in England who may present the bill to him at the expiration of a certain time: *A* then sells to *B* this bill, and receiving £1000 for it, has no longer any claim upon *C*. *B* now sends this bill to *D*, and *D* uses it as a bill for £1000, until the expiration of the time named on the bill, when the money is paid by *C*. Thus these four merchants have been able to have commercial dealings to the amount of £1000 each, without any coin having left either country.

Of course the value of this bill for £1000 depends entirely upon the ability of *C* to *meet* it, that is, to pay the money at the expiration of the time agreed upon in the bill: and we often find that *C*, who was considered able to pay at the time he gave the bill, has become a bankrupt before the time of payment; hence the loss falls on *D*. And this explains the reason why, in a commercial country like England, the failure of one merchant, or firm, causes others to fail: for, in the case above, *D* may also be liable for bills as well as *C*; and if not able to obtain the money which he expected from *C*, may himself become a bankrupt; and so in turn cause other merchants the same loss which he is himself suffering from the failure of *C*.

Now, there are thousands of merchants in the situation of *A* and *B* in America, and similarly of *C* and *D* in England. Hence there are, as a general rule, merchants wanting to buy bills, and others wanting to sell them, in both countries: and what has been said concerning England and America is true with respect to any two countries which export to and import from one another.

These purchases and sales of bills are, for the reason mentioned in (160), conducted through the medium of Bill-brokers.

169. We have just now been supposing that the two countries have imported and exported goods to an equal amount from one another. But suppose that goods had been sent from America to England to a greater amount than from England to America, for instance, to the extent of £1,000,000, then more bills to that amount would go from England to America than from America to England. The merchants in America have in this case plenty of bills, and of course want money for them: but as there are more persons wishing to sell bills than to buy, therefore the bills fetch a lower price: on the contrary, as bills in England are not so plentiful, there are more buyers than sellers, and the bills fetch a high

price than usual. When this increased price exceeds the cost of insurance and loss of interest upon coin sent over to America, the English merchant prefers sending coin instead of paying the increased rate for a bill: and by successive exportations of bullion, the balance of £1,000,000, which was against us, will be paid off: but this increased price of a bill, (which swallows up the profits of business,) or the alternative of paying in coin, causes merchants to be slow in importing until our exports have increased and helped to restore the balance. Here, also, whatever has been said concerning England and America, is of course equally applicable to any two countries which have commercial transactions with each other. In those countries, as South America, where gold and silver are amongst their principal productions, bullion is as much a regular article of export, as woollen or cotton goods would be from England.

170. If the price of a bill in England, entitling the holder to receive gold in a foreign country, be less than the usual course of exchange, the exchange is said to be in favour of England, because there is then no need to export gold.

The following is an example of the quotation of the rate of exchange in the public prints. It is taken from the *Times* of July 18, 1860.

The course of exchange at New York on London for bills at 60 days' sight is $109\frac{3}{4}$ to $109\frac{7}{8}$ p. c.; which, when compared with the Mint par between the two countries [$109\frac{23}{40}$] shows that the exchange is slightly in favour of England; and after making allowance for charges of transport and difference of interest, the present rate leaves a small profit on the importation of gold from the United States.

Def. The standard rate of exchange between any two countries is termed the *Par of Exchange*, or the *Arbitration Price*: but, as alluded to in (166), is not always the same as the *Course* of Exchange. Also, a *Bill on London* means a paper entitling the holder to obtain gold in London, to the value of the amount mentioned in the bill.

Arbitration is called *Simple* or *Compound*, according as there are three or more places concerned.

Ex. V. Bills on Amsterdam, bought in London at 12 florins 15 cents per £ sterling, are sold in Paris at $57\frac{1}{2}$ florins for 120 francs: what is the rate of Exchange between London and Paris?

My object here is to express £1 in terms of francs; hence I express £1 in terms of florins, and florins in terms of francs, and thereby obtain the value of £1 in francs. Working fractionally, I have

$$£1 = 12 \text{ florins } 15 \text{ cents} = 12{\cdot}15 \text{ florins},$$

$$\text{also, } 57\tfrac{1}{2} \text{ flor.} = 120 \text{ francs},$$

$$\text{therefore } 1 \text{ flor.} = \frac{120}{57\frac{1}{2}} \text{ francs},$$

$$\text{and therefore } £1 = 12{\cdot}15 \text{ florins} = 12{\cdot}15 \times \frac{120}{57\frac{1}{2}} \text{ francs}$$

$$= 25 \text{ francs } 35\tfrac{15}{23} \text{ cents}.$$

The following Ex. involves three countries, and therefore three equations.

Ex. VI. A bill upon Hamburgh is bought at 13 marks $10\frac{1}{4}$ schillings per £ sterling, then sold at Amsterdam at $35\frac{3}{4}$ florins per 40 marks: if the proceeds are then remitted to Paris in French bills at $57\frac{1}{4}$ florins per 120 francs, what rate of exchange is there between London and Paris?

$$£1 = 13 \text{ m. } 10\tfrac{1}{4} \text{ sch.} = 13\frac{10\frac{1}{4}}{16} \text{ marks} = 13\tfrac{41}{64} \text{ m.},$$

$$40 \text{ mks.} = 35\tfrac{3}{4} \text{ flor.}, \qquad \text{or } 1 \text{ mk.} = \frac{35\frac{3}{4}}{40} \text{ flor.},$$

$$57\tfrac{1}{4} \text{ flor.} = 120 \text{ francs}, \qquad \text{or } 1 \text{ flor.} = \frac{120}{57\frac{1}{4}} \text{ francs};$$

$$\therefore £1 = 13\tfrac{41}{64} \times \frac{35\frac{3}{4}}{40} \times \frac{120}{57\frac{1}{4}} \text{ francs}$$

$$= 25 \text{ francs } 55\,\tfrac{1405}{3664} \text{ cents.}^*$$

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* The process by which a connection is established between the first and last terms, as for instance, between pounds sterling and francs, is sometimes termed the *Chain Rule*, because these extremities are joined, as it were, by successive links. And the results which we have just investigated, might have been written down at once, by taking as the left-hand side of the equation, the coin whose equivalent is desired,—and for the right-hand side, a frⁿ, of which the numʳ consists of the product of all the successive denominations contained in the question,—and the denʳ, of all the remaining coins of the same kind as the numʳ, with the exception of the last or required coin.

The subjoined is another example of the mode of reporting the exchanges, and is taken from the *Times* of March 22nd, 1848, under the head "Money Market and City Intelligence." It contains an exaggerated view of the general relation between the prices of gold at London and Paris,—because it refers to a time of revolution and panic; but the principle is the same.

> "The last quotation of gold at Paris was about 30 per mille premium, which would give an exchange of 25·91. The last quotation of short bills on London being 26·50, the price of gold would appear to be about $2\frac{1}{4}$ per cent. higher in London than in Paris."

Here a comparison is instituted between the *nominal* rate of exchange of £1 for francs, and the *real* rate at the particular time mentioned. Thus, if 1000 represent the price of gold in Paris when £1 sterling is worth about 25·16 francs*, what will be the value of this £1 in francs when 1030 represents the price of gold in Paris? We have, of course, this statement:

$$1000 : 1030 :: 25{\cdot}16 \text{ francs},$$

$$\text{the fourth term} = 25{\cdot}91 \text{ francs},$$

that is, £1 will in Paris produce 25·91 francs. But a person in Paris wishing to buy a bill on London, entitling him to receive £1 in gold, must give 26·50 francs. The difference, about ·6 francs, is the amount by which £1 sterling in London is dearer than in Paris.

$$\text{Also, since } 26{\cdot}50 \text{ francs} = £1,$$

$$\text{therefore 1 franc} = \frac{1}{26{\cdot}5}£;$$

$$\text{and this extra price of } £1 = {\cdot}6 \text{ fr.} = {\cdot}6 \times \frac{1}{26{\cdot}5}£;$$

$$\text{and therefore the extra price of } £100 = 100 \times {\cdot}6 \times \frac{1}{26{\cdot}5}£$$

$$= \frac{60}{26{\cdot}5}£ = \frac{12}{5{\cdot}3}£ = 2{\cdot}26\ldots£$$

$$\text{or } 2\tfrac{1}{4}£ \text{ nearly},$$

that is, the price of gold in London is $2\frac{1}{4}$ per cent. greater than in Paris; or a bill which would entitle a person to receive £100 in gold in London would cost £$102\frac{1}{4}$ in Paris.

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* See Appendix, Art. *Exchange.*

Exs. 58.

FRANCE AND ENGLAND.

The course of Exchange between France and England is generally about 25 fr. 30 cents for £1 sterling. 100 cents = 1 franc.

1. Exchange £350 for francs, at 25 fr. 50 cts. per £ sterling.
2. ,, £75 10*s*. ,, 25·35½ ,,
3. ,, £425 15*s*. 6*d*. 26·5 ,,
4. ,, 9349·90 francs for pounds, at 25·90 per £ sterling.
5. ,, 21475·12 ,, 25·75 ,,
6. ,, 1875·5 ,, 26·5 ,,

HAMBURGH AND ENGLAND.

13 marks 8 schillings = £1 English. 16 schillings = 1 mark.

m. sch.
7. Exchange £425 for marks, at 13 12 for £1.
8. ,, £375 10*s*. ,, 13 9¼ ,,
9. ,, £87½ ,, 13 8 ,,
m. sch.
10. ,, 1000 marks for pounds sterling, at 13 8 for £1.
11. ,, 8754 mks. 15 sch. ,, 13 10½ ,,
12. ,, 3537·45 mks. ,, 13 11 ,,

THE UNITED STATES AND BRITISH NORTH AMERICA.

1 Dollar = 4*s*. 6*d*.; also sterling money generally bears a premium of 9 per cent.; *i. e.* £100 sterling can be exchanged for as many dollars as will amount to £109; the exact par of exchange is $109\frac{23}{45}$.

13. Exchange £100 for dollars when Eng. money has prem. of 8 p. c.
14. ,, £425 10*s*. ,, ,, 10 ,,
15. ,, £1256 5*s*. ,, ,, 9½ ,,
16. ,, 1000 dollars for Eng. money, when the prem. is 8 ,,
17. ,, 3225 ,, ,, 7½ ,,

EAST INDIES. 1 Rupee about 23¾*d*.

18. ,, £500 for rupees, at 23½*d*. per rupee.
19. ,, 7500 rupees for sterling money, at 24¼*d*.

20. Exchange 2420 rupees for francs, the course of exchange being 40 rupees for 94 francs.

21. Change 1750 marks of Hamburgh for florins at Amsterdam, at the rate of 135 marks for 120 florins.

22. Exchange 3700 francs for Hamburgh marks, at the rate of $187\frac{1}{2}$ francs for 100 marks.

ARBITRATIONS OF EXCHANGE.

23. If the Exchange between London and Amsterdam be $11\frac{3}{4}$ florins per £ sterling, and between Amsterdam and Paris be at the rate of $55\frac{1}{2}$ florins per 115 francs; find the rate of exchange between London and Paris.

24. Bills on Paris, bought at the rate of 25 francs 35 cents per £ sterling, are sold in Lisbon at 190 rees per franc; what rate of exchange is there between London and Lisbon?

25. The exchange between London and Hamburgh is 13 mks. 10 sch. per £ sterling; between Hamburgh and Paris is 150 marks for 275 francs; what exchange does that give between London and Paris?

26. If the exchange between Amsterdam and Hamburgh be at $11\frac{3}{4}$ flors. for $13\frac{1}{2}$ marks, between Amsterdam and Genoa be at $10\frac{2}{3}$ flors. for 25 lire, between Genoa and Portugal be 5 lire for 800 rees, between Lisbon and London be 1 milree or 1000 rees for *54d.*, what exchange does this give between London and Hamburgh? and what difference will there be, between remitting £500 from London to Hamburgh by this circular route, and sending it direct, at an exchange of 13 mrks. 11 schgs. per £ sterling?

27. The premium on gold at Paris is $5\frac{1}{2}$ per mille, which gives an exchange of 25·29; if the quoted exchange of Paris on London be $25{\cdot}22\frac{1}{2}$, shew that gold is 0·26 per cent. dearer in Paris than in London.

Exs. 59. M.

1. A debt of £144 *7s. 6d.* was paid in an equal number of guineas, half-guineas, and seven shilling pieces: required the number.

2. A peck of flour gives 20lbs. of bread; how much land would grow corn enough for $1\frac{1}{2}$ millions of people for a week, at $4\frac{1}{2}$ quarters to the acre, and $1\frac{1}{2}$lbs. of bread for each person per day?

3. The population of a town rises 1 per cent. for 3 years successively; if at the beginning of the 3 yrs. the population were 1 million, what would it be at the end?

4. Tea bought at 1*s.* $10\frac{1}{2}$*d.* per lb. pays a duty of 2*s.* $2\frac{1}{4}$*d.* per lb., what per centage of the whole cost is the taxation?

5. The $3\frac{1}{2}$ per cents. are reduced to $3\frac{1}{4}$ per cents.; 150 millions of stock are so converted: but the holders of 6 millions dissent; if they are paid off while the stock is at $97\frac{1}{2}$, how much will the nation gain or lose in the first year?

6. An estate of 270 acres is bequeathed to three tenants, to be divided in proportion to their rents, which are £180, £120, and £60; how must the land be divided?

7. Find a fourth proportional to $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{16}$; and to ·05, ·015, and ·0075.

8. Find the difference between the Simple and Compound Interest of £520 for 2 yrs. at 5 p. c., where the interest is paid half-yearly.

9. *A*, *B*, and *C* enter into partnership; *A* puts in £100 for 6 months, *B* puts in £200 for 4 months, and *C* £100 for 15 months; divide a profit of £150 equitably.

10. At what rate must I sell an article which cost 50*s.*, so as to gain 15 per cent.?

11. Find the purchase money of £1500 stock in the 3 per cents., at $88\frac{1}{2}$, including $\frac{1}{8}$ per cent. commission.

12. At what rate per cent. will £100 double itself in 8 years, S. Int.?

13. Find the annual income from a legacy of £5000 Stock in the $3\frac{1}{2}$ per cents., after paying the legacy duty of 10 per cent.

14. Prove that the sum of the fractions $2\frac{1}{13}$ and $\dfrac{2}{1\frac{4}{9}}$ is equal to 5 times their difference.

N.

1. Show by a simple Ex., that Mult^{n} is nothing more than a shortened mode of Addition.

2. What would be the length of an acre of ground, if its breadth were 40 yards?

3. Find the value of 4·05 × ·000012; also of 4·05 ÷ ·00012; and prove both results by vulgar fractions.

4. What does £25 15*s.* amount to, when taken ·125 times?

5. If the means of a proportion be 9 and 16, and one of the extremes be 56, what is the other extreme?

6. A person bequeathed £5000 to be divided amongst three persons, in the proportions of 3, 5, and 7; find their respective shares.

7. Explain what fractions produce terminating, and what produce non-terminating decimals. Give Exs. of each.

8. Find the exact difference between ·127*s.* and $\cdot 12\dot{7}$*s.*; and express the result as the fraction of a crown.

9. What sum of money will in 5 years amount to £411 5*s.* at $3\frac{1}{2}$ p. c. Simple Interest?

10. Find the amount of insurance upon £12500, at 4*s.* 6*d.* per cent.

11. What is the present value of £463 10*s.* due in 8 months, allowing $4\frac{1}{2}$ per cent.?

12. How many yards at 6*s.* $7\frac{1}{2}$*d.* per yd. must be given in exchange for 105 yds. at 3*s.* 4*d.* and 375 at 4*s.* $10\frac{1}{2}$*d.*?

13. Explain the meaning of the terms *directly* and *inversely* proportional; and give Exs. of questions illustrating each expression.

14. What decimal multiplied by $\frac{2}{3}$ of $\frac{7}{8}$ of $3\cdot\dot{3}$ will become 17?

15. How many years' purchase should be paid for property, so as to receive $6\frac{3}{4}$ per cent.?

O.

1. Find the Simple Interest of £500 for 5 years at $3\frac{1}{2}$ per cent.

2. What is the number of cubic yds. in 1438790 cubic inches?

3. How often must the sum of 2*s.* $6\frac{1}{4}$*d.*, 3*s.* $9\frac{1}{2}$*d.* and 18*s.* $8\frac{1}{4}$*d.* be repeated, to make £100?

4. If 10 francs be worth 6 florins, and 75 florins be equivalent to 4 moidores; how many francs must be given for 16 moidores?

5. Find the frac^{n} which being multiplied by $\frac{3}{4}$ of $\frac{7}{5}$ of $2\frac{2}{11}$ gives a product$=1$.

6. The interest on a railway share is $3\frac{3}{4}$ per cent.; what is the market value of each share and of the entire line, if money be worth 5 per cent., and the amount paid in dividends be £150,000?

7. A sum of money has doubled itself in 17 years at Simple Interest; what is the rate per cent.?

8. I sell an article for 15*s.* 9*d.*, and by so doing gain 17 per cent.; what was its prime cost?

9. An estate which brings $3\frac{1}{2}$ per cent. lets for £546; what was the purchase money?

10. Divide £70 amongst three persons, whose shares shall be in the ratio of the numbers $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$.

11. Shew whether it is better to invest in the 3 per cents. at 89, or the $3\frac{1}{2}$ per cents. at 94. What difference would there be, if £10000 stock were held?

12. Compare as vulgar fractions $\cdot025 \times \cdot07$, $11\cdot035 \times \cdot0008$, and $\cdot1\dot{9} \times \cdot00\dot{3}$.

13. A person mixes 25 bushels of wheat at 4*s.* 9*d.*, 36 at 5*s.* 6*d.*, and 15 at 6*s.* 6*d.*; what must be the selling price per bushel of the mixture, to gain 10 per cent. on the above prices?

14. Exchange £1050 for francs, at 25 fr. 50 cts. per £ sterling.

15. Exchange 9062 fr. $62\frac{1}{2}$ cts. for pounds sterling, at 25 fr. 35 cts. for £1.

P.

1. Find a fourth proportional to 1, ⅓, ¼; also a third proportional to 12 and 25.

2. Express $\frac{14\frac{3}{4}}{\cdot 025}$ as a simple ratio.

3. I lend £175 for 6 months when money is worth 7 per cent.; for what time ought I to be able to borrow £250, when money is worth but 4½ per cent.?

4. Explain the difference between Interest and Discount; and find the true discount of a bill of £65 13*s*. at 4 months, drawn Oct. 4th, and discounted Nov. 26th, at 5 per cent.

5. At what rate of interest would £350 amount to £389 7*s*. 6*d*. in 3 yrs., at Simple Interest?

6. Goods bought at £2 4*s*. 6*d*. are sold at £2 18*s*. 9*d*.; required the profit per cent.

7. A railway share, originally costing £50, has paid a half-yearly dividend of £1 10*s*.; what will be my rate of interest, if the share cost me £55½?

8. What will be the first year's expense of an insurance on £1500 at a premium of £2 13*s*. 10*d*. per cent., and a stamp duty of £3? Find the per centage, including the stamp.

9. If the price of 500 bricks, of which the length, breadth, and thickness are 12, 4½, and 3 inches respectively, be 12*s*. 6*d*.; how many shall I obtain for the same money, if the dimensions be 15, 6, and 4 inches?

10. Find the prices of investment in the 3, 3½, and 4 p. c. Stocks, when they each pay 3¼ per cent.

11. Distribute the sum of 1000 guineas in the ratio of 1, ⅓, ⅕.

12. The ratio of the invoice price to the net price is 11 : 8; what per centage has been thrown off as discount from the invoice price?

13. Exchange £415 10*s*. for dollars at 4*s*. 6*d*., when English money bears a premium of 7½ p. c.

14. Exchange 3500 dollars for sterling money, when the premium on English money is 10½ per cent.

15. The exchange between London and Paris is 25·5 francs per £ sterling; between Paris and Amsterdam is 117 francs for 55 florins; between Amsterdam and Hamburgh is 11 florins for 13 marks; what is the exchange between London and Hamburgh?

AREA AND VOLUME.

DUODECIMALS.

171. If any line be taken, as *ab*, and upon it a square, *abcd*, be described, this figure may be called the square of *ab*. And if we take *ab* as a unit of length, viz. 1 inch, 1 foot, &c., then *abcd* is called 1 square inch, 1 square foot, &c. We have here, therefore, inches, feet, &c. of length, or *linear* inches; and inches, &c. of surface, or *superficial* inches, or square inches.

Fig. 1.

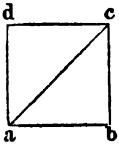
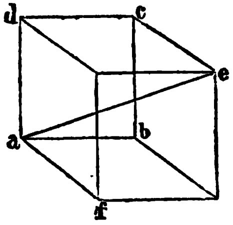


Fig. 2.



172. In like manner, if upon *ab*, the annexed figure *abcdef* be described, having six sides, or surfaces, each equal to *abcd*, it is called a *cube*: and, as before, if *ab* be taken as representing 1 inch, 1 foot, &c., this figure will represent 1 cubic inch, 1 cubic foot, &c.

We have now, therefore, three kinds of units of measurement, viz. linear, or common inches; square, or superficial inches; and cubic, or solid inches. Also, these three units are said to contain 1, 2, and 3 dimensions respectively. For example, the floor of a room, having the dimensions of length and breadth, is of the same nature as a square; and any such surface is called an *Area*: but a cistern of water, having the three dimensions of length, breadth, and depth, is of the same nature as a cube. The quantity contained by such a cistern or similar figure is termed its *Volume*, or solid content.

DEF. A surface which is so enclosed with lines, that any two which meet in a point are perpendicular to one another, is called *rectangular*.

OBS. Any surface bounded by straight lines may be denoted by two letters placed at its opposite corners; and any solid contained by such surfaces may be so denoted: but the two letters employed should not be joined together by a line. Thus, in Fig. 1, I should say the area *bd*, not *ac*; and in Fig. 2, I should say, the volume *fc*, and not *ae*.

DEF. The lines *ac*, *ae*, are called *diagonals*.

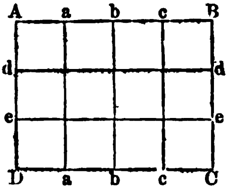
173. I have now to explain how to find the area of surfaces and the volume of solids: but as I do not propose to enter upon mensuration generally, I shall merely treat of rectangular surfaces, as squares and oblongs; and of solids, the surfaces of which are also rectangular.

174. Quantities which can be accurately represented by numbers, whether whole or fractional, are called *commensurable;* and those which cannot be so represented, are called *incommensurable.* Thus it will be shown under the head of "Ratio," in the Appendix, that if in Fig. 1 the length of *ab* be represented by 1, *ac* cannot be accurately represented by any number whatever, whether whole or mixed. The quantity used for the value of *ac* is 1·4142...... and this being nearly the true value, is termed its *approximate* value.

175. We may here state as a fact, that the area of any rectangular surface is found by multiplying the numbers which represent its length and breadth; also that the volume of a solid, bounded by rectangular surfaces, is found by multiplying together the numbers representing the length, breadth, and depth or thickness. The above statements are

true, whether the lines bounding the area or volume be commensurable or not; but they cannot be proved to be universally true without the aid of geometry. We shall, however, give an Ex. illustrating the correctness in each case, choosing of course only commensurable numbers.

Fig. 3.



176. Let $ABCD$ be a rectangular figure whose sides, AB and AD, meeting in the point A, and called *adjacent* sides, contain an exact number of units,—viz. $AB = 4$ inches, $AD = 3$ inches; let the opposite sides, AB, DC, be divided into four equal parts in *a*, *b*, *c*, and the sides AD, BC, be divided into three equal parts, in *d*, *e*; let the lines *aa*, *bb*, *cc*, *dd*, *ee*, be drawn: the figure will be divided into equal squares, each of which has for its side 1 linear inch, as Aa, and is therefore a square inch. Also, counting vertically, there are three rows of squares, because $AD = 3$ inches; and each row contains four squares, because $AB = 4$ inches; so that there are in all 12 squares, *i. e.* 3 × 4 squares: hence we see that the number of square inches in the area $ABCD =$ the product of the number of linear inches in the two adjacent sides, AB, AD.

Def. The lines *aa*, *bb*, which are drawn so that they are at an equal distance from one another, are called *parallel* lines; so also they are said to be *parallel* to AD or BC.

Hence, if I wished to draw lines, as *aa*, *bb*, or *cc*, in the above figure, I should say—through *a*, *b*, *c*, draw lines parallel to AD, or BC. So also, *dd*, *ee*, are drawn parallel to AB, or DC.

If AD had been $= AB$, then the figure would have been a square, and the number of square inches in it would be

4×4, or 4^2 (called 4 *squared*), $= 16$; *i. e.* the number of square inches in any square figure whose side is expressed in linear inches, is found by multiplying the number contained in the side by itself. Hence the second power of a number is called its square, because it represents the area of a square figure, the side of which is the number itself.

We can now show how the numbers mentioned in what is called "Square Measure," are obtained.

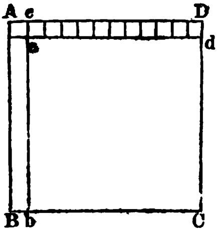
For if 12 linear inches = 1 linear foot;
therefore 12 in. × 12 in. = 144 square inches = 1 square foot.
So also, since 3 linear feet = 1 linear yard,
3 ft. × 3 ft. = 9 square feet = 1 square yard.

Again, $5\frac{1}{2}$ linear yards = 1 linear pole, or perch;
therefore $5\frac{1}{2}$ yds. × $5\frac{1}{2}$ yds. = $30\frac{1}{4}$ sq. yds. = 1 sq. perch;

and these square perches, yards, &c., are the quantities always made use of in measuring land; for no amount of perches in *length* could make up an acre, which consists of *surface*.

177. In measuring surfaces, such as square feet of timber, many arithmeticians have called the twelfth part of a square foot, as *Ad* or *Ab*, an *inch*, and a twelfth part of *Ad*, viz. *Aa*, a *part*: but since a pupil is taught in "Square Measure" that *Aa* is an inch, and that 144 such inches make up a square foot; it is clearly absurd to call *Ad* an *inch*, seeing that 12 such make up a square foot: we shall, therefore, confine the name *inch* to *Aa*, or any quantity equal to it. But the divisions of *AC* have been, and may be, called superficial *primes*, *seconds*, *thirds*, &c.; where it is to be remembered, that a prime is

FIG. 4.



$\frac{1}{12}$th of a square foot, and each succeeding denomination is $\frac{1}{12}$th of the one preceding it. So also, linear inches are sometimes called primes, and twelfths of an inch, seconds.

The square inch, *Aa*, might also be divided, exactly as we have divided *AC*; and its twelfth part would be called a *third*, and be similar in shape, though not in size, to *Ab* or *Ad*; and the twelfth part of this *third* would be called a *fourth*, and be similar in shape to *Aa*.

Also, observing Fig. 4, we learn that

1 foot × 1 foot = 1 sq. foot, as *AC*.

1 foot × 1 inch = 1 sup. prime, as *Ab*.

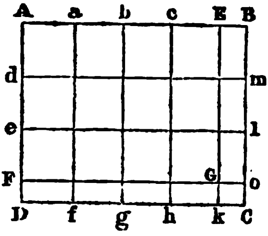
1 inch × 1 inch = 1 sq. inch, or sup. second, as *Aa*.

178. It must be distinctly noticed that we do not multiply together concrete quantities, as feet by feet, inches by inches, &c.; we merely multiply the *number* of feet or inches in the length by the *number* in the breadth; and then we observe that the number obtained in the product is equal to the number of superficial units, primes, &c., in the area contained by the above length and breadth. The same remark applies to the multiplication of three dimensions, in finding the volume of a solid.

Attention to this fact, that concrete quantities cannot be multiplied together, will save persons from the absurdity of attempting to multiply pounds, shillings, or pence, by pounds, shillings, &c. I can multiply 5*s*. by the number 5, and the product is 25*s*.; but if I attempt to multiply 5*s*. by 5*s*., I know of no quantity which can correspond to such a product. And in considering (74) it must be noticed, that though in the practice of Rule of Three I appear to be multiplying the third term by the second, each of which is generally a concrete quantity, and then dividing by the first term, which is also generally a concrete quantity; yet, since the concrete multiplier and divisor are of the same kind, the result is that I have merely multiplied by a fraction, *i. e.* by an abstract number, generally fractional; and the common process has been, that I have multiplied the third term by the numerator, and divided by the denominator of this fraction.

179. We can now find the area of any rectangular figure contained by commensurable lines.

FIG. 5.



Let *ADCB* be a rectangular figure, of which the side $AB = 4$ ft. 6 in., and the adjacent side $AD = 3$ ft. 3 in. Let *AB* be divided into feet in the points *a*, *b*, *c*, *E*, and *AD* so divided in the points *d*, *e*, *F*. Draw *af*, *bg*, *ch*, *Ek*, parallel to *AD*, or *BC*: and draw *dm*, *el*, *Fo* parallel to *AB* or *DC*, according to (176). Let *Ek*, *Fo*, intersect in *G*. Then the whole area $AC = AG + BG + DG + CG$.

Also, from (177) we have

	sq. ft.	sup. pr.	sq. in.
$AG = AE \times AF = 4$ ft. × 3 ft. = 12 sq. ft. =	12	0	0
$BG = EB \times EG = 6$ in. × 3 ft. = 18 sup. pr. =	1	6	0
$DG = FG \times FD = 4$ ft. × 3 in. = 12 sup. pr. =	1	0	0
$GC = Go \times Gk = 6$ in. × 3 in. = 18 sq. in. =		1	6
Area *AC* =	14	7	6

180. Observing these four results, we may place the work in a condensed form, as annexed.

Since our proposal is to find the product of 4 ft. 6 in. by 3 ft. 3 in., we place one number under the other nearly as in Compound Multiplication, and commence multiplying the lowest denomination in the upper line by that in the lower line, and so on through the multiplier. I give the operations which are required to perform the work mentally, observing that each product as it is formed can be reduced to the next higher denomination by dividing it by 12. I commence at the right-hand and proceed thus: 3 × 6 = 18, which, divided by 12, gives 1 to carry to column (II.), and 6 to put down:

III.	II.	I.
	ft.	′in.
	4	6
	3	3
1	1	6
13	6	
14	7	6

$3 \times 4 = 12$ for column (II.), which, with the 1 carried, is 13, and divided by 12 gives 1 to carry to (III.), and 1 to set down in (II.) Again, commencing with the multiplier 3 feet, I have $3 \times 6 = 18$ in (II.), which divided by 12 gives 1 to carry to (III.), and a remainder 6 for (II.): lastly, $3 \times 4 = 12$ for (III.), which, with the 1 to carry, becomes 13. Adding the two rows, I have the result 14 sq. ft. 7 sup. primes, 6 sq. in.; or, bringing the primes to square inches, I have 14 sq. feet 90 sq. inches, as in (179).

Comparing the operations of this article with those of the last, I notice that the steps which produced the first product of the multiplication sum in (180) are the same as the third and fourth in (179); and those which produced the second product are the same as the first and second in that article. Hence in Exs. similar to the one now worked, we need not draw any figure to insure the correctness of the work obtained by the multiplication in (180). This mode of working is called *Cross Multiplication*, and sometimes *Duodecimals.* The latter name is given in consequence of the work of such an Ex. being precisely the same as in Simple Multiplication; provided that, in working from right to left, we take every figure as having *twelve* times the value which it would have one place to the right, instead of *ten* times that value, as in common numbers.

That the area $= 4\frac{1}{2} \times 3\frac{1}{4}$ sq. ft. may also be thus shown; but to pupils unaccustomed to follow a demonstration, the above proof appealing to the eye is often more satisfactory.

Let the sides *AD, AB,* be divided into fourths of a foot, *i. e.* of the linear unit: the former ($4\frac{1}{2}$, or $\frac{18}{4}$) will contain 18 such parts, and the latter ($3\frac{1}{4}$ or $\frac{13}{4}$) will contain 13; and if parallels are drawn through these several points of division, the whole rectangle *ABCD* will be cut up into a set of small squares, each of which is the 16th part of the square unit, (1 sq. ft.); and the number of the small squares will plainly be 18 taken 13 times, or 18×13. Hence the area, expressed in terms of the square unit, (1 sq. ft.), will be $\frac{18 \times 13}{16} = \frac{18}{4} \times \frac{13}{4} = 4\frac{1}{2} \times 3\frac{1}{4} =$ the product of the numbers representing the two adjacent sides.

This area $=\frac{9}{2}\times\frac{13}{4}$ sq. ft. $=\frac{117}{8}$ sq. ft.

$=14\frac{5}{8}$ sq. ft. $=14$ sq. ft. 90 sq. inches, as before.

181. Next, let one or both of the adjacent sides of any rectangular figure, whose area is required, consist of feet, inches, and twelfths of an inch; we must then further divide a square inch as we divided a square foot in (177): and we then learn that

1 inch × $\frac{1}{12}$ inch $=\frac{1}{12}$th of a sq. inch, or 1 super. third;
$\frac{1}{12}$ inch × $\frac{1}{12}$ inch $=\frac{1}{144}$ super. inch, or 1 superficial fourth.

We can now, without further explanation, follow the work of the accompanying Ex.

Find the area of a rectangular floor, whereof the length is 9 feet 4 inches 7 seconds, and the breadth 5 feet 6 inches 4 seconds.

$$\begin{array}{rrrrr}
 & & \text{ft.} & \text{pr.} & \text{sec.} \\
 & & 9 & 4 & 7 \\
 & & \underline{5} & \underline{6} & \underline{4} \\
 & 3 & 1 & 6 & 4 \\
4 & 8 & 3 & 6 & \\
46 & 10 & 11 & & \\
\hline
51 & 10 & 4 & 0 & 4 \\
\hline\hline
\end{array}$$

The highest denomination, 51, in this product has principally been obtained from the multiplication of 5 feet and 9 feet, and is therefore square feet; and the remaining denominations are superficial primes, seconds, thirds, fourths: and the whole answer is written 51 sq. feet 10 sup. pr. 4 sec. (or sq. in.) 0 thirds, 4 fourths; or, neglecting the fourths, 51 sq. feet, 124 sq. inches.

Exs. 60.

1. Multiply 4 ft. 7 in. by 8 ft. 3 in.
2. „ 13 ft. 5 in. by 27 ft. 9 in.
3. „ 2 ft. 6 in. 4 sec. by 11 ft. 3 in.
4. „ 18 ft. 4 in. by 3 ft. 6 in. 9 sec.
5. „ 7 ft. 2 in. 5 sec. by 11 ft. 3 in. 4 sec.
6. „ 15 ft. 0 in. 7 sec. by 13 feet. 0 in. 11 sec.

7. What is the cost of paving a rectangular area, 20 ft. 6 in. by 4 ft. 3 in., at 30*s.* per square yard?

8. A room is 97 ft. 8 in. in circuit, and 9 ft. 10 in. high; what will the painting come to at 7½*d.* per square yard?

9. How much will remain out of 393 square feet of carpeting, after covering a floor 23 ft. 6 in. long, and 16 ft. 7 in. broad?

10. Find the whole surface of a box, whereof the height, length, and breadth are respectively 3 ft. 2 in., 7 ft. $5\frac{7}{12}$ in., and 4 ft. 8 in. 5 sec.

11. What is the whole surface of a cubical box, the edge of which is 2 ft. 9 in.?

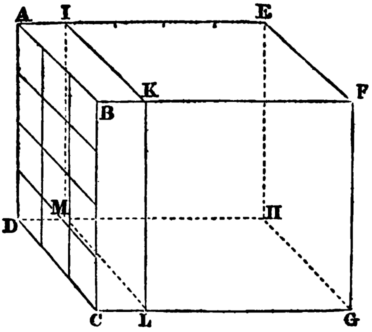
12. A house has 63 windows; 40 of them contain 12 panes, each 20 in. by 16 in.; the remainder contain 9 panes, 16 in. square; find the cost of glazing the whole at 2*s.* 3*d.* per square foot.

13. A Turkey carpet, measuring 11 ft. 6 in. by 9 ft. 8 in., is laid down on the floor of a room measuring 14 ft. by 12 ft. 6 in.; find the quantity of oil-cloth necessary to complete the covering of the floor.

182. We have stated that the length, breadth, and depth of a solid must be multiplied together to obtain its volume; we now proceed to give an Ex. of the truth of this, just as we have already worked one in a question of two dimensions.

First, let all the dimensions consist of whole numbers, viz. 3, 4, 5 inches; and let $ABCGH$ be the solid, where $AE=5$, $AB=3$, $AD=4$.

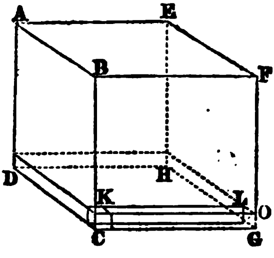
Fig. 6.



Let $IKLM$ be a section of the whole solid by a plane parallel to $ABCD$: then, since $AE=5$, the whole solid may be divided by similar planes into solid blocks, the base of each of which is $ABCD$, and the thickness 1 inch: hence there will be five times as many solid inches in the whole AG, as there are solid inches in AL.

Now, by (176) the area $ABCD = 4$ in. × 3 in. = 12 sq. in: and if upon each of these square inches a solid inch be placed, there will be twelve solid inches in AL. Hence, in the whole AG, *i. e.* in five times AL, there will be 5×12 cubic inches; *i. e.* the volume of the solid $= 5 \times 4 \times 3$ solid inches. Of course, if the three dimensions had been expressed in feet instead of inches, the result would have been $5 \times 4 \times 3$ solid feet.

FIG. 7.



Let now the solid have all its dimensions equal, and each = 1 foot, then the figure is a cubic foot, and its volume = 12 in. × 12 in. × 12 in. = 1728 cub. in., and 12 × 12 × 12, or 12^3 is called 12 *cubed*. Hence we can obtain the numbers exhibited in what is called "Solid Measure."

For 12 in. × 12 in. × 12 in., or 1728 solid inches = 1 solid foot;

3 ft. × 3 ft. × 3 ft., or 27 solid feet = 1 solid yard.

By a classification similar to that of (177), we term a twelfth part of a solid foot, a solid or cubic prime; a twelfth of a prime, a second; and so on: but we must always remember, that though we use but one set of names, we have three kinds of primes, seconds, &c., viz. linear, superficial, and solid; and the various sorts can never be added to or subtracted from each other—for it is evidently impossible to add an area either to a line or a volume.

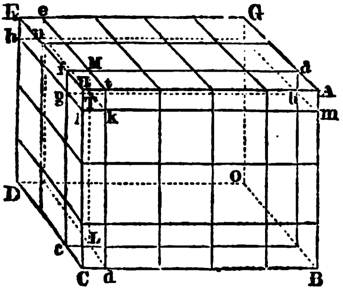
By observing Fig. 7 we learn that

1 sq. foot × 1 linear inch = 1 solid prime, as *DO*.

1 sup. prime × 1 linear inch = 1 solid second, as *CL*.

1 sq. inch or 1 sup. sec. × 1 linear inch = 1 solid third, or solid inch, as *CK*.

FIG. 8.



Let *ABCDEG* or *BE* be a solid contained by rectangular surfaces; let the three adjacent edges, *CB*, *CD*, *CH*, be respectively equal to 4ft. 5in., 2ft. 6in., and 3ft. 4in.; or $4\frac{5}{12}$, $2\frac{1}{2}$, $3\frac{1}{3}$ feet. According to the second method of (180), let each of these adjacent edges be divided into twelfths (12 being the least common denominator of their fractional parts); then the number of twelfths in them will be 53, 30, 40 respectively; and through each of the points of division draw planes parallel to the several faces of the given solid; then the whole volume will evidently be divided into smaller cubes, each having for its edge one-*twelfth* of the linear unit, and the *volume* of which will therefore be $\frac{1}{1728}$th of the former solid unit. Also, the number of these small cubes will by (182) be $53 \times 30 \times 40$; and therefore the volume expressed in terms of the cubic unit, (1 cubic foot,) will be $\frac{53 \times 30 \times 40}{1728}$, or $\frac{53}{12} \times \frac{30}{12} \times \frac{40}{12}$, or $4\frac{5}{12} \times 2\frac{1}{2} \times 3\frac{1}{3}$, which equals the product of the numbers representing the three edges which meet in a point.

By Cross Multiplication, the work will be as follows:

```
         4  5
         2  6
        ------
     2   2  6
     8  10
   -----------
    11   0  6
         3  4
  ------------
  3  8   2  0
 33  1   6
 -------------
 36  9   8  0 or 36 116/144 sol. ft. or 36 29/36 sol. ft.
 =============
```

183. Working fractionally, the volume $= \frac{53}{12} \times \frac{5}{2} \times \frac{10}{3}$ sol. ft.

$= \frac{1325}{36}$ sol. ft. $= 36\frac{29}{36}$ sol. ft.

184. Since length × breadth × depth = volume (I.), therefore length × breadth $= \frac{\text{volume}}{\text{depth}}$; and since length × breadth is of two dimensions, therefore $\frac{\text{volume}}{\text{depth}}$ is of two dimensions, *i. e.* $\frac{\text{3 dimensions}}{\text{1 dimension}}$ gives two dimensions.

So also, from (I.) length $= \dfrac{\text{volume}}{\text{breadth} \times \text{depth}}$; *i. e.* $\dfrac{\text{3 dimen.}}{\text{2 dimen.}}$ gives one dimension: and any fraction in which numerator and denominator are of the same dimension, is of no dimensions, or an abstract number. (See Art. 67.)

185. The following Exs., are amongst the most useful of those in which the measurement of surfaces and solids occurs.*

Ex. I. Find the number of acres in a rectangular field, of which the length is 35 chains 72 links, and the breadth 24 chains 8 links.

To understand this question, a pupil must know that large pieces of land are measured by means of a chain called *Gunter's Chain*, which is four poles, or 22 yards, in length, and is divided into 100 equal parts, called *Links*.

Also, an acre is equal to a rectangular surface of which the length is 40 poles, or 10 chains, and breadth 4 poles, or 1 chain: hence, the area of an acre which is 40 poles long, and 4 poles broad, = 10 chains × 1 chain = 1000 links × 100 links = 100,000 square links. Consequently, if the dimensions of a field be expressed in links, and its area thence be obtained in square links, this value, when divided by 100,000, will be expressed in acres; *i. e.* if five places be pointed off as a decimal, the result will be acres and decimal parts of an acre, which can be reduced to roods and poles. Returning now to the Ex., we have

$$\begin{array}{rr}
\text{Length} = 35 \text{ chains } 72 \text{ links} = & 3572 \text{ links.} \\
\text{Breadth} = 24 \text{ chains } 8 \text{ links} = & \underline{2408} \text{ links} \\
 & 28576\phantom{\text{ links}} \\
 & 142880\phantom{\text{ links}} \\
 & \underline{7144}\phantom{\text{ links}} \\
\text{Area} = \text{acres} & 86{\cdot}01376\phantom{\text{ links}} \\
 & \underline{4}\phantom{\text{ links}} \\
\text{Roods} & {\cdot}05504\phantom{\text{ links}} \\
 & \underline{40}\phantom{\text{ links}} \\
\text{Perches} & \underline{\underline{2{\cdot}20160}}\phantom{\text{ links}}
\end{array}$$

and therefore the field contains 86 a. 0 r. $2\frac{1}{5}$ perches, nearly.

Ex. II. Find how many gallons are contained in a cistern of which the length is 40 inches, breadth 36 inches, and depth 16 inches.

---

* For further information on Mensuration, see Lund's Geometry, Pt. III.

It must here be observed that an imperial gallon is equal to 277·274 cubic inches: hence, when the number of solid inches in any volume is known, the number of gallons which it will contain is found by dividing those solid inches by 277·274. In the above Ex. we therefore have

$$\text{the solid content} = (40 \times 36 \times 16) \text{ solid inches};$$

$$\text{and therefore number of gallons contained} = \frac{23040}{277 \cdot 274} = 83, \text{ nearly.}$$

Ex. III. A roof of $27\frac{1}{2}$ feet by $18\frac{3}{4}$ is to be covered with lead weighing 8 lbs. per square foot: what would it cost at the rate of £5 4*s.* for 5 cwt.?

$$\begin{aligned} \text{The area of the roof} &= 27\tfrac{1}{2} \text{ ft.} \times 18\tfrac{3}{4} \text{ ft.} \\ &= \left(\frac{55}{2} \times \frac{75}{4}\right) \text{sq. ft.} \\ &= \frac{4125}{8} \text{ sq. ft.} \end{aligned}$$

$$\text{hence, weight of lead} = \frac{4125}{8} \times 8 \text{ lbs.} = 4125 \text{ lbs.}$$

To find the cost of the lead, we have the following statement:

$$5 \times 112 \text{ lbs.} \quad : \quad 4125 \text{ lbs.} \quad :: \quad £5\ 4s.$$

The fourth term will be found to be £38 6*s.* $0\frac{6}{7}$*d.*

Exs. 61. Form the following products:—

1. 3 ft. 2 in. × 4 ft. 9 in. × 5 ft. 7 in.
2. 15 sq. ft. 73 sq. in. × 2 ft. 6 in.
3. 11 sq. ft. 9 sup. pr. 3 sec. × 9 ft. 7 in.
4. 2 ft. 4 in. × 3 in. × 11 in.

5. Find the capacity of a rectangular cistern, 12 ft. 3 in. long, 5 ft. 7 in. broad, and 2 ft. 11 in. deep.

6. How many bricks, of which the length, breadth, and thickness are 12, 9, 6 inches respectively, will be required to build a wall, whereof the length, height, and thickness are 64, 9, and $1\frac{1}{2}$ feet?

7. What is the price of a block of stone, of which the length, breadth, and thickness are 37 ft. 8 in.; 8 ft.; and 6 ft. 5 in., at 5*s.* 6*d.* per sol. foot?

8. How many square feet of board would be required to make a rectangular box, of which the length, breadth, and depth are respectively $3\frac{1}{2}$ ft., $2\frac{1}{4}$ ft., and 1 ft. $2\frac{5}{12}$ in.?

9. A cubic inch of water weighs 252·458 grains, required the weight (in lbs. Troy) of water in a full cistern $10\frac{1}{2}$ ft. long, $5\frac{1}{4}$ ft. wide, and 11 in. deep.

10. The bottom of a cistern is rectangular, and contains 15 sq. feet, 58 sq. in.; how deep must it be to hold 164 gallons, if a gallon contain $277\frac{1}{4}$ cubic inches?

11. The length of a rectangular field is 25 chains 37 links, and the breadth 17 chains 35 links; find the number of acres contained.

12. Find the number of chains and links in the breadth of a field, whereof the length is 35 chains 15 links, and the area is 45 a. 2 r. $31\frac{1}{4}$ p.

EXTRACTION OF ROOTS.

186. We have already seen (92) that a number, by being successively multiplied by itself, is said to be raised to a power; and the order of the power, whether it be second, third, fourth, &c., depends upon the number of times the original number is to be repeated. This process is termed *Involution;* and the reverse process of obtaining the original number from the power is called *Evolution.* The original number is called, with respect to its power, the *Root;* and this evolution is also termed *Extraction of Roots.*

The Square Root of any number or quantity is that number or quantity, which when squared, that is, multiplied by itself, will produce the original quantity.

The Cube Root is that which when cubed, or multiplied by itself *twice,* will produce the original quantity.

Thus, since 5×5, or 5^2, or 5 *squared* (176), as it is called, $= 25$; 5 is called the Square Root of 25. Similarly, since $6 \times 6 \times 6$, or 6^3, or 6 *cubed* $(182) = 216$, then 216 is termed the cube or third power of 6, and 6 the cube root or third root of 216.

The sign $\sqrt{\ }$ is used to express the operation of extracting a root; and a small figure, placed thus $\sqrt[3]{\ }$, shows what root [here the *third* root] is to be extracted: but the figure is generally omitted when the sign refers to the square root;

and the sign $\sqrt{\ }$ itself indicates extraction of the Square Root.

187. A number or quantity which is thus formed by the squaring, cubing, &c. of any number, is called a complete or *perfect* square, cube, &c.; and therefore the square, cube, &c. root, of such a power can be exactly extracted: but any number, as 20, which lies between two complete squares, 16 and 25, cannot have its root obtained exactly. But since 16 has a root 4, and 25 has a root 5, therefore the root of 20 will be between 4 and 5, or its value will be approximately expressed by the mixed decimal 4·472...; and similarly of every number which lies between two complete squares. So also every number which lies between two complete cubes must have its cube root expressed in a decimal form.

188. If a number be raised to a power, and then the same root of that power be extracted, we shall of course obtain the original quantity; therefore $\sqrt{3^2} = 3$; $\sqrt[3]{5^3} = 5$.

Also, since the square root of a given number has been defined to be that number which, when multiplied by itself, will produce the given number, therefore $\sqrt{2} \times \sqrt{2} = 2$: so also, $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$. Such quantities as $\sqrt{2}$, $\sqrt[3]{2}$, are called *Surds*, or *Irrational* quantities.

OBS. Since fourth, fifth, &c. powers of numbers can be obtained, of course fourth, fifth, &c. roots must also exist; but we shall here confine ourselves principally to the finding of square and cube roots.

What has been said of the raising of whole numbers to powers is also true of fractional quantities, whether vulgar, or commensurable decimals. Thus, since $\frac{2}{3} \times \frac{2}{3}$, or $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$; therefore $\sqrt{\frac{4}{9}} = \frac{2}{3}$, which equals $\frac{\sqrt{4}}{\sqrt{9}}$; *i. e.* the square root of a fr^n must be obtained by taking the square root of numerator and denominator. In like manner, since $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$; therefore $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$, which equals $\frac{\sqrt[3]{8}}{\sqrt[3]{27}}$; *i. e.* the cube root of a fraction will be found by taking the cube root of both numerator and denominator. If the root of a mixed number be required, it must be reduced to an improper fraction or a decimal. Thus:

$$\sqrt{1\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4} = 1\tfrac{1}{4}.$$

We shall presently show how to extract the square and cube roots of decimals as well as of whole numbers.

SQUARE ROOT.

189. Before proceeding to investigate a rule for the extraction of the Square Root, we must remember the following:

Digits,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Squares,	1,	4,	9,	16,	25,	36,	49,	64,	81.

From what has been said, it is plain that the square of 1, is 1; of 10, is 100; of 100, is 10,000, &c. &c.

Therefore the square root of		1	is	1
,,	,,	100	,,	10
,,	,,	10,000	,,	100
,,	,,	1,000,000	,,	1000 &c. &c.

Hence, if a number lie between 1 and 100, *i. e.* have not more than *two* figures, its square root is between 1 and 10, or has not more than *one* figure.

If between 100 and 10,000, *i. e.* have not more than *four* figures, the root is between 10 and 100, or has not more than *two* figures.

If between 10,000 and 1,000,000, *i. e.* have not more than *six* figures, the square root is between 100 and 1,000, or has not more than *three* figures: and so on.

So that if over every alternate figure of any number, beginning at the units' place, a point be placed, the number of points will show the number of figures in the square root. For example, 3757 has more than two figures, and not more than *four*, or lies between 100 and 10,000, and therefore its root lies between 10 and 100, *i. e.* has not more than *two* figures, and the number must therefore be thus pointed, $3\dot{7}5\dot{7}$; so also $\dot{1}3\dot{4}5\dot{9}$ is correctly pointed, for there will be three figures in the square root. The divisions which are formed by these points are called *periods*. The periods in $\dot{1}3\dot{4}5\dot{9}$ are 1, 34, and 59.

Also, since the square of ·1 is ·01, therefore
the square root of ·01 is ·1
So also, of ·0001 is ·01
" " ·000,001 is ·001 &c. &c.

It hence appears that in the square of any decimal an even number of decimal places will always be found; and that if there be an odd number in any proposed Ex., the number of places must be made even, by appending a cipher, which cannot alter the value of the decimal. We then point every alternate figure of a decimal, from left to right, beginning with the second figure from the units' place, so that the last figure will always be pointed, in decimals as well as in whole numbers.

190. The Rule for the Extraction of the Square Root is derived from an algebraical operation, wherein a complete square is taken, and a process is then contrived by which the root, which is already known, can be deduced from the complete square.

I shall therefore proceed, contrary to my usual method, to give a Rule for extracting the Square Root, without any more explanation than is necessary for the mere working of Exs.: and the arithmetical illustration of the above algebraical process, which I shall afterwards give, may be read or omitted at the reader's pleasure.

RULE. Divide the given number into periods. Find the greatest square number which is not greater than the first period; subtract it from that period, and place the root of the number in the place of the quotient in Long Division.

To the remainder, after subtraction, bring down two figures or one period, and consider the whole as a dividend.

For a divisor, double the root, and try how often it is contained in the dividend, except the last figure: the figure thus obtained by division place in the root, and annex it to the divisor.

Multiply this whole divisor by the last figure in the root, and subtract this *subtrahend* from the dividend.

Bring down another period; find a fresh divisor by adding the last figure of the former divisor to that divisor, and proceed exactly as before.

OBS. It will be found that on dividing by the *incomplete* divisor—especially when the early figures in the root are small, and the latter ones large—there will result a quotient larger than the one which must be taken. The reason of this will be explained hereafter.

I will work one Ex. which gives no remainder, *i. e.* where the given number is a perfect square, viz. 81796.

A pupil will find himself less liable to make mistakes in bringing down the periods, if in actual working he points off as I have done below.

```
      8'17'96' (286
      4
     ------
 48) 417
      384
     ------
566)   3396
       3396
     ======
```

The greatest square number not greater than the first period, 8, is 4; I therefore subtract 4 from the 8, and place in the quotient, 2, the root of the 4: to the remainder, 4, I bring down the second period, 17, and consider 417 as my dividend. For a divisor, I double the 2 in the root: and on dividing 41 by this divisor 4, I obtain a quotient 10; but upon trial I find that 8 is the largest quotient that can be employed. I place the 8 in the root, as well as at the right of the 4 in the divisor, making a *complete* divisor 48: multiplying this 48 by the 8, I have a subtrahend 384; the remainder, after subtraction, is 33, and with the third period 96 gives a new dividend 3396: adding the last figure 8 in the divisor 48, to that 48, I have, as new partial divisor, 56, which goes six times in 339: placing the 6 in the root and in the divisor, and multiplying the whole divisor 566 by the 6, I have a subtrahend 3396, which, upon subtraction, leaves no remainder. The square root is therefore 286.

Since most of the numbers we meet with are not perfect squares, we can obtain only approximate roots of such numbers; and the operation is carried to about three places of decimals in the root, requiring of course, six places in the number. If the given number be an integer, or have not so many as six decimal places, the number must be made up by appending ciphers.

Ex. II. Find the square root of 876·535.

```
         8'76'·53'50'00' (29·606...
         4
    49) 476
        441
   586)  3553
         3516
 59206)    375000
           355236
            19764
```

I append three ciphers, and point according to (189). When the third divisor, 592, is obtained, the quotient is 0; I therefore place the 0, as usual, in the root and in the divisor, then bring down another period, and proceed as before. Since there were two periods in the integral part of the given number, there will be two places of integers in the root, and the decimal point must be put after the 29.

191. I will now illustrate the algebraical process mentioned in (190), as far as can be done in arithmetic.

Let the square number 169 be taken, the root of which is 13. If this number and its root be expressed in the required algebraic form, they will be respectively written 100+60+9, and 10+3.

```
         100+60+9 (10+3
         100
  20+3)      60+9
             60+9
```

Putting in the place of the quotient the 10+3, which is known to be the root, I observe that the first part of the root, viz. 10, is the square root of 100, the first part of the number: I may therefore consider that the square root of the 100 gives the 10, and the remainder 60+9 is to furnish the 3. Observing the former part of the remainder, viz. 60, I notice, that if it were divided by twice the 10 in the root, I should obtain the required number, 3: I therefore make 20, *i. e.* twice the first figure in the root, my divisor. Also, since there must be no remainder, I must have as subtrahend 60+9; and since the 9 is the square of 3, therefore, if I append the 3 to the 20 in the divisor by the sign (+), and multiply the whole divisor 20+3 by 3, I shall obtain a subtrahend equal to the dividend 60+9, and therefore shall have no remainder, as was required.

An algebraical process similar to the above would *prove* that this method of procuring a divisor and subtrahend would *always* succeed in bringing no remainder, *i. e.* in obtaining the exact root of a complete square. But this Ex. of course only *illustrates* the method of proof, and shows how the Rule is algebraically deduced. The operation, when condensed into an arithmetical form, stands thus:

$$\begin{array}{r|l} & 1'69'\ (13 \\ & 1 \\ \hline 23) & 69 \\ & \underline{\underline{69}} \end{array}$$

192. I will work one more Ex. which will show how an error in the quotient may arise in dividing by an incomplete divisor.

$(27)^2=729$, which may be put into the algebraic form $400+280+49$: and the root is $20+7$.

And this, when condensed, is

$$\begin{array}{r|l} & 400+280+49\ (20+7 \\ & 400 \\ \hline 40+7) & 280+49 \\ & \underline{\underline{280+49}} \end{array} \qquad\qquad \begin{array}{r|l} & 7'29'\ (27 \\ & 4 \\ \hline 47) & 329 \\ & \underline{\underline{329}} \end{array}$$

OBS. Since to divide 280 or 320 by 40 is the same as to divide 28 or 32 by 4, I may therefore call 4 the divisor, when 28 or 32 is the dividend, and consider 40 as the divisor, when 280 or 320 is the dividend.

If this Ex. be worked in the usual manner, it will be found that the incomplete divisor 4 will go eight times in 32, whereas the real quotient is found to be only 7. By observing the algebraic method, I learn, that when the dividend 329 is separated into its parts 280 + 49, the true quotient is obtained by dividing only the former part, 280, by the 40, or 28 by 4: but in the arithmetical operation I cannot see how much of 329 is the former part which will give me a correct quotient; and therefore I have to divide the whole 329 by 40, or 32 by 4, and so run the risk of an error. If the second part of the dividend 49 had been less than 40, the quotient would at once have appeared to be 7, and no error arisen. Hence the larger the second part of the dividend (as 49) is, when compared with the incomplete divisor (as 40), the greater will be the error: and since the 4 in the divisor was obtained from the *former* part of the quotient, viz. 2, and the 49 from the *latter* part, viz. 7, the error will be the greatest when the earlier figures of the root are small, and the latter are large.

This method of proof is not limited to numbers, the roots of which consist but of two figures: but any attempt to extend the illustration would be very cumbrous without the use of algebra.

Exs. 62.

Extract the Square Root of each of the following numbers:—

1. 729
2. 11025
3. 8264446281
4. 2832489
5. $27\frac{9}{16}$
6. $371\frac{46}{49}$
7. $1241\frac{152}{289}$
8. $12122\frac{1}{100}$
9. 10·5625
10. 197·4025
11. 36·343
12. ·002401

193. We have seen (176) that the area of a square is obtained by squaring any one of its sides; hence the number in the side of a square is found by extracting the square root of the number which represents the area of the square.

The number which represents the area of a square figure may not always be a complete square: for instance, if in Fig. 1, p. 171, a square be described with sides equal to *ac* or $\sqrt{2}$, then its area would be $(\sqrt{2})^2 = 2$, which is not a perfect square. When, therefore, the number representing the area of a square is not itself a perfect square, the number representing the side will be incommensurable or irrational.

Quantities which are not perfect squares, cubes, &c. may be made so by multiplying them by certain factors. For, in order that a number may be a complete square, each of its factors must be contained 2, 4, 6, &c. times, *i.e.* some multiple of *two* times; to be a complete cube, each must be contained 3, 6, 9, &c. times, *i.e.* some multiple of *three* times; and so on for higher powers. For example, the number 144 will be found $=2^4 \times 3^2$, where the indices 4 and 2 are multiples of 2, and therefore 144 is a perfect *square.* So also $1728 = 2^6 \times 3^3$, where the indices are multiples of 3, and therefore 1728 is a perfect *cube.*

Hence, if I resolve any number into its factors, I can tell by inspection whether it be a perfect square, cube, &c.; and if not, what additional factors must be introduced into it to make it so. Thus $20 = 2^2 \times 5$; and since the index of the 5 is not a multiple of 2, it must be made so, by introducing an additional 5; it then becomes $2^2 \times 5^2$, or 100, a perfect square. Again, to make 48 a perfect cube, I observe that $48 = 2^4 \times 3$; and in order that the indices may be multiples of 3, this must be changed into $2^6 \times 3^3$, and therefore be multiplied by $2^2 \times 3^2$, *i.e.* by 36; and we then shall have $48 \times 36 = 1728$, a perfect cube.

In obtaining the square root of a fractional quantity, we may extract the root of numerator and denominator, if the denominator be a complete square, as in (188).

$$\text{Thus, } \sqrt{\frac{729}{1225}} = \frac{27}{35};$$

or, taking the case of a mixed number,

$$\sqrt{4\frac{49}{144}} = \sqrt{\frac{625}{144}} = \frac{25}{12} = 2\tfrac{1}{12}.$$

But if the den^{r} be not a perfect square, it is better to reduce the fraction to a decimal, and then extract the root. For example, if I have to extract the square root of $27\frac{27}{128}$, then, since $27\frac{27}{128} = 27{\cdot}2109375$, a single extraction will suffice to give the answer at once; whereas, trying the former method, I should have

$$\sqrt{27\tfrac{27}{128}}=\sqrt{\frac{3483}{128}}=\frac{59{\cdot}01\ldots\ldots}{11{\cdot}31\ldots\ldots},$$

and to obtain the result, I must extract *two* roots, and perform an operation in Long Division.

Again, we may make the den^r a complete square, as described in the preceding paragraph; and then, since the root of the den^r will be known by the very process of completing the square, it will be necessary to extract only the root of the num^r. Thus, taking the above example, and observing that a factor 2 will make the 128 become 256…, or 16^2, we have

$$\sqrt{27\tfrac{27}{128}}=\sqrt{\frac{3483}{128}}=\sqrt{\frac{6966}{256}}=\frac{83{\cdot}462\ldots}{16}=5{\cdot}216\ldots.$$

In finding the root of a circulating decimal, it will sometimes happen that the equivalent vulgar fraction will have both numerator and denominator complete squares, and we can then readily extract the root. Thus:

$$\sqrt{1{\cdot}\dot{7}}=\sqrt{1\tfrac{7}{9}}=\sqrt{\tfrac{16}{9}}=\tfrac{4}{3}=1\tfrac{1}{3},\text{ or }1{\cdot}\dot{3}.$$

But as this is rarely the case, we must generally find the approximate root by the usual method, though it will not be a recurring decimal.

194. Sometimes a fraction involving a surd may be in its lowest terms, but yet not in a form the most convenient for finding its value. Thus $\frac{1}{\sqrt{2}}$ is in the lowest terms; but multiplying numerator and denominator by $\sqrt{2}$, it becomes $\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{2}$, or $\frac{1}{2}\sqrt{2}$. And we shall find that this fraction is more simple than the former one. For, since $\sqrt{2}=1{\cdot}4142\ldots$

therefore $\frac{1}{\sqrt{2}} = \frac{1}{1{\cdot}4142\ldots}$, which will require a Long Division sum to bring out its value; but $\frac{1}{2}\sqrt{2} = \frac{1}{2}(1{\cdot}4142\ldots) = {\cdot}7071..$ by mere Short Division. Hence $\frac{1}{2}\sqrt{2}$ is more simple than $\frac{1}{\sqrt{2}}$.

And as a general rule, all quantities involving surds are in their simplest form, when the surds are in the numerator and not in the denominator.

The following Exs. illustrate the operations of the last few pages.

Ex. I. A square field contains 15 a. 2 r. 20 p. Find its side in chains.

15 a. 2 r. 20 p. = 2500 sq. poles; therefore the side of a field containing 2500 sq. poles = 50 linear poles

$= \frac{50}{4}$ linear chains (since 4 poles = 1 chain)

$= 12\frac{1}{2}$ linear chains.

Ex. II. Two acres of land are to be cut from a rectangular field, of which the breadth is 2 chains 50 links, by a line parallel to it. Find the length of the plot.

1 acre = 40 p. × 4 poles = 10 chains × 1 chain = 10 sq. chains; and length × breadth = 2 acres; or, since the breadth is $2\frac{1}{2}$ chains, therefore length × $2\frac{1}{2}$ chains = 2 acres = 2 × 10 sq. chains = 20 sq. chains.

Hence length $= \frac{20 \text{ sq. chains}}{2\frac{1}{2} \text{ linear chains}}$

$= \overset{4}{\cancel{20}} \times \frac{2}{\cancel{5}}$ linear chains (184)

$= 8$ linear chains.

Exs. 63. Find the value (to 4 places of decimals) of

1. $\sqrt{3}$ 2. $\frac{1}{\sqrt{3}}$ 3. $\sqrt{6{\cdot}2\dot{4}\dot{9}}$ 4. $\sqrt{15{\cdot}\dot{3}}$

5. Find the side of a square field, whose area is equal to that of a rectangle 1800 yds. by 800 yds.

6. The sides of 2 squares are 15 ft. and 25 ft.; find the side of another which shall equal the sum of the two former.

7. A rectangular field measures 64 ft. in length by 48 ft. in breadth; what is the length of the diagonal?*

8. A square field contains 3 a. 3 r. $2\frac{58}{121}$ p.; find the length of its side in yards.

9. The painting of a square area, at $7\frac{1}{2}d$. per square yard, comes to £22 15*s*. $7\frac{1}{2}d$.; find the length of the side of the square.

10. Two sides of a triangle are respectively 236·25 and 243·75 ft., also the altitude is $2\frac{2}{5}$ ft.; find the length of the base*.

CUBE ROOT.

195. The following numbers must be first remembered:

Digits,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729:

and it will be observed that no two of these cubes end with the same digit: hence, if a given number be a perfect cube, the last figure in its root may be known. For instance, if a cube number end in 3, its cube root ends in 7: if the number end in 2, its root ends in 8. Recollection of this fact is often of service.

According to the method of (181), we learn that the cube root

of 1 is 1
" 1,000 " 10
" 1,000,000 " 100 &c. &c.

hence, if a number lie between 1 and 1000, *i. e.* have not more than three figures, its cube root is between 1 and 10, or has not more than one figure.

If the number be between 1000 and 1,000,000, *i. e.* have not more than six figures, its root has not more than two figures: and if the number have not more than nine figures, its root has not more than three figures.

So that if over every third figure, beginning at the units' place, a point be placed, the number of points will show the number of figures in the cube root.

Similarly, since the cube of ·1 is ·001
therefore the cube root of ·001 " ·1
So also,........................ ·000,001 " ·01
" " " ·000,000,001 " ·001 &c. &c.

It hence appears that the number of places in the cube of any decimal must always be some multiple of 3: and if the number of places in any decimal, of which we have to find the cube root, be not a multiple of 3, it must be made so by appending ciphers: we therefore commence pointing at the units' place, and point every third figure to the left over the integers; and to the right over the decimals, if there be any.

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* See Appendix, Note to *Art. Ratio.*

The following Rule, called Horner's method, though at first difficult for a learner, yet has the merit of exhibiting all the work in a very convenient form, especially when the given number is large. But the proof of it would be out of place here, as it involves some knowledge of the Theory of Equations.

Rule. I. Divide the given number into periods. Find the greatest cube number which is not greater than the first period. Subtract it from that period; place the root of the number in the quotient, and form a dividend as in Square Root.

II. To the left of the number, and at some distance from it, place two columns (A) and (B). Under (A) insert three times the root, and under (B) three times its square. Annex one cipher to (A), and two ciphers to (B), and with (B) as divisor, find the next figure in the quotient.

III. Add this figure to (A); also add to (B) the product of (A) by the last figure in (A); and multiply the sum by the last figure in the root, to form a subtrahend; subtract this from the dividend, and bring down one period.

IV. Under (A) place twice its units' figure, and under (B) the square of that figure. Find the sum of the last *two* lines in (A), and of the last *three* in (B), and again annex one cipher to (A) and two to (B). With (B) as divisor, find another figure in the root, and proceed to form another subtrahend, as in (III.) of this Rule.

Obs. Here, as in Square Root, the divisor often gives upon trial too large a quotient. But in working examples in Algebra by it, there is no such liability to error.

196. The following Example will be found to exemplify this Rule; and the lines have been written widely, to indicate the successive steps from (A) to (B), and from (B) to the subtrahends.

A		B		21'717'639' (279
60				8
7		1200 = total divisor)		13717
67 × 7	=	469		
14		1669 × 7	=	11683
810		49		
9		218700		)2034639
819 × 9	=	7371		
		226071 × 9	=	2034639

Since the first figure in the root is 2, I place 3×2, or 6, under (A), and 3×2^2, or 12, under (B), annex one cipher to (A) and two ciphers to (B). Using this 1200 as a divisor, I have 7 as the second figure in the root, which I add to (A), making 67, and to (B) I add the product of this 67 by 7, viz. 469, making 1669: I multiply this 1669 by the same 7, making a subtrahend 11683, and obtain a dividend 2034639. I now place under (A) twice its last figure 7, and under (B) I place 7^2 or 49; I now add the last *two* lines in (A), making 81, and the last *three* lines in (B), viz. 49, 1669, and 469, obtaining 2187; then annex one cipher to (A), two ciphers to (B), and making this 218700 in (B) as a divisor, I obtain a figure 9 in the root. I now add this 9 to the 810 in (A), and multiply the whole of it, or 819, by 9, making 7371, then add it to (B), making 226071; lastly, multiply this by the same 9, placing the product as a subtrahend. If there had been any more periods, I should have commenced forming a fresh divisor, as in (IV.).

197. The observations which were made upon the extraction of the Square Root of Fractions in (193) apply also to a similar use of the Cube Root.

Just as the number in the side of an area of square form is found by extracting the square root of the number representing the area: so the number in the edge of a volume of cubic form is found by extracting the cube root of the number which represents the volume.

I will work one or two Exs. to illustrate the application of Cube Root.

Ex. IV. A beam is 6 ft. 9 in. long, 2 ft. 3 in. broad, and 9 in. thick; required the side of a cube of equal capacity.

Working fractionally, we have

$$\text{solid content of beam} = 6\tfrac{3}{4}\text{ ft.} \times 2\tfrac{1}{4}\text{ ft.} \times \tfrac{3}{4}\text{ ft.}$$

$$= \left(\frac{27}{4} \times \frac{9}{4} \times \frac{3}{4}\right)\text{ sol. ft.} = \frac{729}{64}\text{ sol. ft.};$$

$$\text{therefore edge of the cube} = \sqrt[3]{\frac{729}{64}\text{ sol. ft.}} = \frac{9}{4}\text{ lin. feet} = 2\tfrac{1}{4}\text{ linear ft.}$$

Ex. V. Find the area of any one of the six surfaces of a cube containing 11 cubic feet, 675 cubic inches.

To find the area of this surface, I must find the edge of the cube, and square it. Reducing the cubic feet to inches, I have

$$\text{volume of cube} = 19683\text{ cubic inches};$$

$$\text{therefore edge of the cube} = \sqrt[3]{19683} = 27\text{ linear inches} = 2\text{ ft. }3\text{ in.};$$

$$\text{and area of the side in sq. ft.} = (2\tfrac{1}{4})^2 = \left(\frac{9}{4}\right)^2 = \frac{81}{16} = 5\tfrac{1}{16}$$

$$= 5\text{ sq. ft. }0\text{ pr. }9\text{ sq. in.}$$

Ex. VI. Assuming that the volume of any cylinder equals its length × area of its base, find the value of this area in a cylindrical wire 50 feet long, and made out of a square inch plate of metal ·05 inches in thickness.

Here, since area × length = whole volume of metal

$$= 1\text{ sq. inch} \times \text{thickness of the plate};$$

$$\text{or, since area} \times 50\text{ ft.} = 1\text{ sq. inch} \times \cdot 05\text{ inches};$$

$$\text{therefore area} = \frac{1\text{ sq. in.} \times \cdot 05\text{ in.}}{50 \times 12\text{ in.}}$$

$$= \frac{\cdot 05}{50 \times 12}\text{ sq. inches}$$

$$\left(\begin{matrix}\text{multiplying numerator}\\ \text{and denominator by 100}\end{matrix}\right) = \frac{5}{5000 \times 12}\text{ sq. inches}$$

$$= \frac{1}{12000}\text{ sq. inches.}$$

Exs. 64. Extract the Cube Root of each of the following numbers:—

1. 42875
2. 970299
3. 6539203
4. 32798729601
5. $42\frac{7}{8}$
6. $1367\frac{631}{1000}$
7. $2345\frac{22}{343}$
8. $423987\frac{457}{512}$
9. 77·854483
10. 284890·312
11. 1334·633301
12. $\dfrac{\cdot000405224}{\cdot064}$.

Find the value (to 2 places of decimals) of

13. $\sqrt[3]{15}$ 14. $\sqrt[3]{155}$ 15. $\sqrt[3]{342\cdot\dot{9}}$ 16. $\sqrt[3]{\dfrac{1}{15}}$

17. Find the edge of a cubical box containing 13824 solid inches.

18. A cistern is 72 ft. long, 24 ft. broad, and 27 ft. deep: find the edge of a cubical cistern of the same content.

19. Find the length of the edge of a cube which contains 94 yds. 14 ft. 1088 inches.

20. Find the whole surface of a cube which contains 15 solid feet and 1080 solid inches.

TO EXTRACT ANY ROOT WHATEVER.

198. Upon the same principle as the Cube Root Rule given above, is the following Rule for the extraction of any root whatever.

I. Divide the given number into periods, each containing as many figures as the index of the root to be extracted. Find the root of the first period, and place it as a quotient.

II. Form, at equal distances from each other, columns, called A, B, C, &c. equal in number to the above index; and consider the first period in the given number as the head of the last column. Under that to the left, place the figure in the root; under the next, the square of that figure; under the next, the cube, and so on. The highest power of this figure will fall under the first period; subtract it from that period, and form a new dividend as usual.

III. To find a trial divisor:—Under (A) again place the figure in the root; multiply (mentally) the sum by this root; but place the product under (B), not under (A); add the two lines in (B), multiply the sum by the root, and place the product under (C); proceed in like manner to the *last column* which stands before the proposed number.

Commence at (A) precisely the same process, and continue it to the *last column but one;* again continue the same process, dropping a column each time, till only the first one is employed. Now annex *one* cipher to (A), *two* to (B), *three* to (C), and so on; and make the result of the column, which precedes the proposed number, a trial divisor of the dividend, thus obtaining another figure in the root. The subtrahend will be obtained presently.

IV. The next trial divisor is obtained by forming a series of products with the new figure, of precisely the same kind as those obtained from the former figure. Also, in forming the first row of these new products, the last one to the right will be the subtrahend from the lately formed dividend, and will after subtraction furnish a new dividend as usual.

Ex. To extract the fifth root of 60466176.

A	B	C	D	E
				60466176 (36
3	9	27	81	243
3	18	81	324	
6	27	108	4050000	) 36166176
3	27	162	1977696	36166176
9	54	270000	6027696	
3	36	59616		
12	9000	329616		
3	936			
150	9936			
6				
156				

In the above Ex. it may be seen that the first figure in the root, viz. 3, must be found by trial, as in Cube Root. Under the five columns, A, B, C, D, E, there have been placed, 3, 3^2 or 9, 3^3 or 27, 3^4 or 81, 3^5 or 243, this last power being the subtrahend from the first period. Also, this figure 3 has been added to (A) four times; each successive result has been multiplied by this figure 3, and the products added to (B). In like manner, of the three results similarly formed in (B), the first two have been multiplied by the root figure, and the products carried to (C). The result of the first addition alone in (C) has been multiplied and carried to (D). The results of the columns are now 15, 90, 270, 405, and with the ciphers added become 150, 9000, 270000, 4050000, which last, taken as a trial divisor, gives 6 as the next figure in the root.

To find a subtrahend, we proceed through the first steps which would be necessary if we had a third period, and were obtaining another trial divisor.

Shew that 25 is the fifth root of 9765625, and 27 the sixth root of 387420489.

Exs. 65. Q.

1. Explain the process of finding the area of an oblong, the sides whereof contain any number of feet and inches.

2. Explain by figures the nature of the products when feet are multiplied by feet, feet by inches, inches by inches, &c. down to twelfths of an inch.

3. Find how much money must be paid for £10800 stock at $84\frac{7}{8}$, and 2*s*. 6*d*. per cent. brokerage.

4. A mixture is made of 40 lbs. at 3*s*. 6*d*., 44 lbs. at 3*s*. $10\frac{1}{2}$*d*., and 55 lbs. at 4*s*. 6*d*.; what will be the gain per cent. if the mixture be sold for £39 4*s*.?

5. What is the interest of money invested in the $3\frac{1}{2}$ per cents. at $89\frac{3}{8}$?

6. An inclined plane 3 miles long has a total rise of 212·15 feet; find the rise per yard in decimal parts of an inch.

7. Reduce to a simple fraction the ratio between the sum and difference of these expressions :—

$$\left(\tfrac{3}{2}+\tfrac{2}{3}\right)\times\left(\tfrac{5}{4}+\tfrac{4}{5}\right) \text{ and } \left(\tfrac{3}{2}-\tfrac{2}{3}\right)\times\left(\tfrac{5}{4}-\tfrac{4}{5}\right).$$

8. Form the following product, (17 ft. 4 in.) × (18 ft. 7 in.), by Duodecimals, and by Fractions.

9. Shew that $\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$; which is the simpler of the two, and why?

10. Find the square roots of 6424·0225 and of $\frac{729}{1296}$.

11. What is the solid content of a box, of which the height, length, and breadth are respectively 3 ft. 2 in., 5 ft. 4 in., and 2 ft. 7 in.?

12. What is the true discount on a 4 months' bill for £151 7*s*., drawn on January 1st, and discounted on the 20th of February, at $4\frac{1}{2}$ per cent.?

13. An article which sold for 50 guineas caused a loss of $5\frac{1}{2}$ per cent., what should it have fetched, to produce $7\frac{1}{2}$ per cent. profit?

14. If the rate of exchange between Amsterdam and Paris be 21·1 francs for 10 florins, and between London and Amsterdam be 12·15 florins for £1 sterling, what is the rate of exchange between London and Paris?

R.

1. State two numbers between 1 and 100, such that the first has exact square and cube roots, and the second has exact square and fourth roots.

2. If a cubic foot of air weighs $1\frac{1}{4}$ oz., what is the weight of air contained in a room 25 ft. 6 in. long, 16 ft. broad, and 10 ft. 9 in. high?

3. A cubical block contains 24 ft. 1403 in.; find its edge, and the area of its 6 sides.

4. Find the value of $\sqrt{23\frac{29}{45}}$; and of $\sqrt[3]{168\frac{5}{16}}$ to two places of decimals.

5. What is the extent of surface of a covered box, of which the dimensions are 5 ft. 10 in., 3 ft. 6 in., and 7 ft. 2 in.?

6. If the painting of a room 9 ft. 6 in. high, 15 ft. 3 in. long, and 10 ft. broad, come to £5 7*s.* 6*d.*; what will be the expense of painting another of which the length, breadth, and height are 18 ft. 2 in., 11 ft. 7 in., and 12 ft. 8 in.?

7. If 5000 rupees produce £505 4*s.* 2*d.*, what is the rate of exchange between England and the East Indies?

8. If by selling goods at £2 5*s.* I lose 15 per cent., what would be the loss or gain per cent. if the price were £2 18*s.* 6*d.*?

9. The sides of two square pieces of ground are 360 yds. and 160 yds.; find the value of the latter, if the former be worth £300.

10. Railway shares which were purchased at a premium of 50 per cent. and sold at a discount of $25\frac{1}{2}$ per cent., produce a loss of £7550; how much money was invested?

11. The length of a rectangular field is 12 chains 35 links, and the breadth is 10 chains 75 links; find the number of acres contained.

12. A square field contains 9 a. 0 r. 4 p.; find the length of its side in chains.

13. If $\frac{1}{3}$ of a debt be due in 3 months, $\frac{1}{4}$ in 4 months, $\frac{1}{6}$ in 6 months, and the remainder in 10 months; what is the equated time of payment?

14. Explain the meaning of the term "Fractional Quotient."

S.

1. A person employs 6 men for 5 days at 8 hours each, and 5 women for 6 days at 10 hours, at the respective wages of 4*d.*, and $2\frac{1}{4}$*d.* per hour; how much must he pay them?

2. Find the number of men in a side of a square, if when drawn up in rank and file they number 81 by 36.

3. Reduce to the simplest form $(\frac{2}{3}$ of $\frac{7}{8}$ of $1\frac{7}{11}) \div (2\frac{1}{5}$ of $3\frac{4}{7})$.

4. The discount on £153 due half a year hence is £3, what is the rate of interest?

5. In what time will £750 amount to £918 15*s.* at $4\frac{1}{2}$ per cent. Simple Interest?

6. How many years' purchase should be paid for freehold property to produce $5\frac{1}{4}$ per cent.?

7. What is the value of a perpetual annuity of £60, reckoning money worth 4 per cent.?

8. A testator bequeaths £500 to *A*, £300 to *B*, and £450 to *C*, but his estate only produces £600; find each man's share.

9. A cubic foot of water weighs 1000 oz.; what is the weight of water in a full cistern, the dimensions of which are 7 ft. 6 in., 5 ft. 2 in., and 3 ft.?

10. The mercury in a barometer rises uniformly from 29·15 to 30·73 in 12 days; find the ratio of the daily rise to the average height.

11. A square field has a diagonal 160 yards long; find the area of the field*.

12. If a solid foot of gold weigh 1260 lb. 8 oz., and a solid foot of cork weigh 15 lb. Troy; how much cork will weigh as much as 1 inch of gold?

13. A cubical mass of metal, of which the edge is 3·35 inches in length, is drawn out into a wire, of which the area of a section is ·561125 square inches; find the length of the wire.

14. Shew how to make any number a complete 2nd, 3rd, 4th, &c. power. Ex. Find a multiplier which shall make 75 a perfect cube.

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* See Appendix, Note to Art. *Ratio.*

SCALES OF NOTATION.

199. It will often have been observed that each figure in any number has *ten* times the value that it would have had, if it had been one place to the right. The selection of this number *ten* has arisen from its being the number of fingers on the hands; and hence has arisen the use of the term *digits*, for the Arabic figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Hence 10 is called the Radix of the common Scale* of Notation, and the scale itself is called the *denary* scale, from the Latin distributive numeral DENI, *in tens.*

If any other number be chosen as the radix of a scale, the new scale derives its name from the corresponding Latin term.

Thus, if the radix be	2,	the scale is called the	*Binary.*
,,	3	,,	*Ternary.*
,,	4	,,	*Quaternary.*
,,	5	,,	*Quinary.*
,,	6	,,	*Senary.*
,,	7	,,	*Septenary.*
,,	8	,,	*Octenary.*
,,	9	,,	*Nonary.*
,,	10	,,	*Denary.*
,,	11	,,	*Undenary.*
,,	12	,,	*Duodenary.*

In the two latter scales, since 11 and 12 digits are required, therefore the letters *t* and *e* are used to indicate *ten* and *eleven.* Multiplication in the duodenary scale is really *Duodecimals.*

200. We have to shew how to convert any number in the common scale into another scale; and conversely.

Now it will be easily seen that any number, as 5372, in the common scale, can be written in the following algebraical form,

$$5000+300+70+2, \text{ or } 5\times10^3+3\times10^2+7\times10+2. \qquad \text{(I.)}$$

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* The term Scale has here the meaning of the word SCALA, a ladder, whereby we, as it were, *ascend* from lower to higher numbers by steps of 10.

And if 8 were the radix of the scale in which this said number 5372 was arranged, the digits of which it is composed would be different; and if *a*, *b*, *c*, *d* represented these new digits, which I do not yet know, I should have

$$a\times8^3+b\times8^2+c\times8+d=5372. \qquad \text{(II.)}$$

Now, in series (I.), we may observe that if we divide by the radix 10, we shall have as quotient $5\times10^2+3\times10+7+\frac{2}{10}$; *i. e.* $\frac{2}{10}$ is the frac^{l} part of the quot^{t}, or as we commonly express it, 2 is the rem^{r}; *i. e.* the last digit of a number in scale 10 is found by dividing the entire number by the radix 10. Also, if we continue successively to divide the quot^{t} $5\times10^2+3\times10+7$ by 10, we shall get as rem^{rs} 7, 3, and 5; *i. e.* the successive digits from right to left are obtained by dividing by the radix, until there be no dividend left. So also, if we divide the left-hand side of (II.) by the new radix 8, the quotient is $a\times8^2+b\times8+c+\frac{d}{8}$ or the remainder *d* is the last digit, as 2 was, just above. Now the two sides of equation (II.) have of course precisely the same value, and since when I divide the *left*-hand side of (II.) by 8, I obtain the last digit in scale 8, therefore I shall obtain the same digit by dividing the *right*-hand side 5372 by 8.

$$\begin{array}{llllllll}
8\,\underline{)\,a\times8^3+b\times8^2+c\times8+d} & & & & 8\,\underline{)\,5372} & & & \\
8\,\underline{)\,a\times8^2+b\times8\ +c} & +\frac{d}{8} & \text{or rem}^r & d & 8\,\underline{)\,671} & -\ \frac{4}{8} & \text{or rem}^r & 4 \\
8\,\underline{)\,a\times8\ +b} & +\frac{c}{8} & ,, & c & 8\,\underline{)\,83} & -\ \frac{7}{8} & ,, & 7 \\
8\,\underline{)\,a} & +\frac{b}{8} & ,, & b & 8\,\underline{)10} & -\ \frac{3}{8} & ,, & 3 \\
 & \frac{a}{8} & ,, & a & 8\,\underline{)1} & -\ \frac{2}{8} & ,, & 2 \\
 & & & & \underline{\underline{0}} & -\ \frac{1}{8} & ,, & 1
\end{array}$$

Placing the two quantities in equation (II.) in parallel columns, and dividing both by 8, I obtain as quotients, $a \times 8^2 + b \times 8 + c + \frac{d}{8}$, and $671\frac{4}{8}$. Hence the quotients $a \times 8^2 + b \times 8 + c$, and 671 are equal, and the rem^{rs} d and 4 are equal; *i.e.* the first of the new digits to the right is 4. The second division by 8 gives equal quotients, viz. $a \times 8 + b = 83$, and a rem^{r} $c = 7$. The next gives quotients a and 10 respectively, and a rem^{r} $b = 3$. Hence the values of a, b, c, d, are 10, 3, 7, 4; but since in the scale of 8 there can be no single number $a = 10$, I therefore divide the 10 still further, till I obtain *two* more digits, 2 and 1, in place of the 10, and the entire number of digits will be 1, 2, 3, 7, 4, &c.; and it now appears that there will be 5 figures in the number 5372 when expressed in the scale of 8, and the number will be represented by 12374.

And we can reconvert this number into the denary scale thus; it equals

$$\begin{aligned} &1 \times 8^4 + 2 \times 8^3 + 3 \times 8^2 + 7 \times 8 + 4 \qquad \text{(III.)} \\ &= 4096 + 1024 + 192 + 56 + 4 \\ &= 5372, \text{ as before.} \end{aligned}$$

Hence we learn that if we have to transfer a number from the denary scale to any other scale of notation, we have only to divide the proposed number by the new radix, till there is no integral quotient. The successive remainders will be the new digits, the first one filling the units' place.

The process of reconverting from any of these new scales to the denary is shown in (III.), where the number 12374 in the octenary scale is restored to the denary, giving 5372.

The fact that 5372 in the scale of 10 is equal to 12374 in the scale of 8 is thus briefly exhibited;

$$(5372)_{10} = (12374)_8.$$

201. We have now seen how to convert from the denary scale into any other scale, and from any scale into the denary. But if we wish to change from any one of the new scales into any other one, as for example, from the scale of 5 to that of 7, we must first change from the quinary to the denary, and then from the denary to the septenary.

Ex. Change 34201 from the quinary scale to the septenary.

First, converting $(34201)_5$ into the denary scale, I have

$$\begin{aligned}(34201)_5 &= 3\times 5^4+4\times 5^3+2\times 5^2+0\times 5+1\\ &= 3\times 625+4\times 125+2\times 25+1\\ &= 1875+500+50+1\\ &= (2426)_{10} \qquad\qquad (\text{IV.})\end{aligned}$$

$$\begin{array}{r@{\;}l}
7\,) & \underline{2426}\\
7\,) & \underline{346} - 4\\
7\,) & \underline{49} - 3\\
7\,) & \underline{7} - 0\\
7\,) & \underline{1} - 0\\
 & \underline{\underline{0}} - 1
\end{array}$$

Now, converting 2426 into the septenary scale, according to (200), as in the margin, we have as the new digits, 1, 0, 0, 3, 4, or $(2426)_{10}=(10034)_7$. And this equality may be proved by the reconversion of $(10034)_7$ into the denary scale; thus

$$\begin{aligned}(10034)_7 &= 1\times 7^4+0\times 7^3+0\times 7^2+3\times 7+4\\ &= 2401+21+4\\ &= (2426)_{10}.\end{aligned}$$

Hence, we have $(34201)_5=(2426)_{10}=(10034)_7$.

Exs. 66.

1. Change 345 from the common or denary scale to the binary.
2. ,, 10101 from the binary to the denary.
3. ,, 1375 ,, denary ,, ternary.
4. ,, 89*t*4 ,, undenary ,, quinary.
5. ,, 14328 ,, nonary ,, septenary.
6. ,, 7854 ,, denary ,, duodenary.
7. ,, 3*t*45 ,, undenary ,, binary.
8. ,, 11111 ,, binary ,, denary.
9. ,, 4444 ,, duodenary ,, quinary.
10. ,, 21021 ,, ternary ,, senary.
11. Add together in the senary scale, 25341, 5423, 4021, 13450.
12. Find in the duodenary scale the value of 89*t*36*e*—4567*t*.
13. Express the result of 187*t*56 × *t*4789 in the duodenary scale.
14. Find the quotient of 18763÷456 in the nonary scale.
15. Find the area of a floor 15 ft. 11 in. by 7 ft. 9 in., by multiplying in the duodenary scale.

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202. It is a very good exercise for pupils to be able, without using any specific rules, to work examples which are generally wrought by rules; or, as it is sometimes expressed, to work them according to the principles of common sense, of course employing the four Simple and Compound Rules, and Reduction.

The most favourable examples for this exercise are those generally wrought by the Rules given in Simple and Compound Proportion. But here, as in (133), I will not assume proportionality in the quantities involved.

Ex. I. If 9 men earn £15 10*s*. in 3 months, or 13 weeks, what does each man earn per week?

If in 13 weeks 9 men earn 310*s*.

then, in 13 weeks 1 man earns $\frac{310}{9}$*s*.

or in 1 week $\frac{1}{13}$th of $\frac{310}{9}$*s*.

or $\frac{310}{117}$*s*. or 2*s*. 7$\frac{31}{39}$*d*.

Ex. II. If £240 be paid for bread sufficient to serve 49 persons for 18 months, when wheat is 48*s*. per quarter; how long will £235 find bread for 92 persons, when wheat is 56*s*. per quarter?

At 48*s*. per qr. £240 will serve 49 persons for 18 months,

∴ „ £1 „ 1 person for $\frac{49 \times 18}{240}$ months,

and at 1*s*. per qr. £1 „ 1 person 48 times $\frac{49 \times 18}{240}$ m.

Hence, at 56*s*. per qr. £1 „ 1 person $\frac{48}{56} \times \frac{49 \times 18}{240}$ months;

∴ „ £235 „ 92 persons $\frac{235}{92} \times \frac{48}{56} \times \frac{49 \times 18}{240}$ m.

or, 8$\frac{17}{368}$ months.

This question worked as a Compound Proportion Example, according to (140), would furnish the following statement.

$$\begin{array}{lclcl} \text{£}240 & & \text{£}235 & & \\ 92 \text{ men} & : & 49 \text{ men} & :: & 18 \text{ months.} \\ 56s. & & 48s. & & \end{array}$$

and the fourth term would $= \frac{235 \times 49 \times 48 \times 18}{240 \times 92 \times 56}$ months,

the same as before.

Ex. III. Find the present worth of £169 18*s*. 4*d*., due 15 months hence, at 5 per cent.

Here £100 in 15 months at 5 per cent. produces £6 5*s*., or amounts to £106 5*s*.

i. e. the present worth of £$106\frac{1}{4}$ is £100,

and ∴ of £1 is £$\frac{100}{106\frac{1}{4}}$,

and ∴ of £169 18*s*. 4*d*., or of £$169\frac{11}{12}$, is $\frac{169\frac{11}{12}\times 100}{106\frac{1}{4}}$£,

the result which would be obtained by a statement, such as in (149).

Ex. IV. If 7 men can do a piece of work in $15\frac{1}{2}$ days, in what time will 8 men and 7 boys do the same, reckoning a boy's labour worth $\frac{5}{8}$ that of a man?

$$7 \text{ boys} = \frac{5}{8} \text{ of } 7 \text{ men};$$

$$\therefore 8 \text{ men and } 7 \text{ boys} = \left(8 + \frac{5}{8} \text{ of } 7\right) \text{ men}, = (8 + 4\tfrac{3}{8}) \text{ men} = 12\tfrac{3}{8} \text{ men}.$$

Now, if 7 men do the work in $15\frac{1}{2}$ days,

1 man will do it in $15\frac{1}{2}\times 7$ days,

∴ $12\frac{3}{8}$ men „ in $\frac{15\frac{1}{2}\times 7}{12\frac{3}{8}}$ days,

or, in $\frac{\frac{33}{2}\times 7}{\frac{99}{8}}$ days,

or, in $\frac{\cancel{33}}{\cancel{2}}\times\frac{7}{1}\times\frac{\overset{4}{\cancel{8}}}{\underset{8}{\cancel{99}}}$ days, or $\frac{28}{3}$ days, or $9\frac{1}{3}$ days.

EXS. 66 A.

EASY MISCELLANEOUS EXAMPLES.

1. In £100 how many piastres at 8*s*. 7*d*. each?

2. What will be the expense of carpeting a room $23\frac{3}{4}$ feet long, and $16\frac{1}{4}$ feet wide, at 2*s*. 9*d*. per square yard?

3. Express the sum of the fractions $1\frac{7}{8}$, $\frac{3}{8}$ of $\frac{41}{34}$, and $\frac{4}{5\frac{1}{10}}$ by a fraction in its lowest terms.

4. What sum will discharge a debt of £56 5*s*., an abatement of 15 per cent. being made for present payment?

5. What will the carpeting of a room $16\frac{1}{2}$ feet square amount to, at 4*s*. $10\frac{1}{2}$*d*. a square yard?

6. If a brick be 9 inches long, 4 wide, and 3 thick, how many will be required to build a wall 1 foot 10 inches thick, 100 yards long, and $4\frac{1}{2}$ yards high?

7. An estate of 750 acres, which pays an average rent of £1 12*s*. 6*d*. an acre, is burdened with a mortgage of £2500, for which interest is paid at the rate of 4 per cent. per annum; what is the clear rental of the estate?

8. Reduce $\frac{3}{5}$ of $(6\frac{3}{8}+2\frac{1}{3})$, and $\dfrac{2\frac{1}{4}-\frac{2}{3}\text{ of }1\frac{5}{8}}{\frac{1}{5}\text{ of }3\frac{1}{3}+\frac{13}{36}}$ to their simplest forms.

9. *A*, who travels $3\frac{1}{4}$ miles an hour, starts $2\frac{1}{3}$ hours before *B*, who travels the same road at the rate of $4\frac{1}{3}$ miles an hour; when will *B* overtake *A*?

10. Extract the square root of 81144064 and of $\frac{2}{3}$ to three places of decimals.

11. Add together $\frac{5}{12}\times\frac{11}{15}$ of a guinea, $\frac{5}{8}$ of a shilling, and $\frac{1}{12}$ of half-a-crown.

12. Reduce $\frac{3}{5}$ of $7\frac{1}{2}$ of $16\frac{1}{2}$ yards to the fraction and decimal of a furlong.

13. Find the breadth of a room, the length of which is $17\frac{1}{4}$ feet, and the area $250\frac{5}{8}$ sq. feet.

14. How much money must be invested in the 3 per cents. at 84 to produce an annual income of £150?

15. What fraction of £3 10*s*. is £2 5*s*. 6*d*.? and reduce the result to a decimal.

16. How many cubic feet and inches are there in a solid, whose breadth is 9 feet 3 inches, length 11 feet 5 inches, and height 3 feet 2 inches?

17. Find the cost of 2627 sacks at *7s.* 8½*d.* per sack.

18. Find the price of a piece of wainscoting 7 ft. 9 in. long, and 9 ft. 3 inches broad, at 2*s.* per ft.

19. What ready money will discharge a debt of £85 due 5 months hence, if an abatement be made at the rate of 13*s.* 4*d.* per cent. per month?

20. What is land let for per acre, when the rent received for 56 acres, 3 roods, 17 perches, is £85 5*s.* 8¼*d.*?

21. What length of carpet ¾ yds. wide will cover a room whose length is 42 ft. 5 in. and breadth 31¾ ft.?

22. How much money must a person invest in the 3 per cents. at 90⅜ to produce a half-yearly income of £50?

23. A cubic foot of water weighs 1000 ounces avoirdupois; what is the weight of a cubic inch in ounces?

24. Goods are bought for £193 12*s.* and sold for £217 16*s.*; what was the gain per cent.?

25. If 6 horses in 2 days of 12 hours each plough 17 acres, how many roods will 2 horses plough in 8 hours?

26. Find the G. C. M. of 36·595 and 57·980.

27. Extract the square roots of 202500 and ·000576.

28. If the length of the year be taken at 365¼ days, instead of 365·242264 days, which is the true length, what will the error amount to in a century?

29. Bought a quantity of goods for £150 ready money, and sold them again for £200 payable ¾ of a year hence, what was the gain in ready money, allowing discount at 4½ per cent.?

30. If discount be treated as interest, find the gain in the last question.

31. If the expense of 3 persons on a tour of 5 months be £315, how many persons will spend £378 during a tour of 9 months?

32. If 2 tons, 3 cwt. 3 qrs. be bought for £112 10*s.*, at what rate per cwt. must the article be sold, so as to clear 50 per cent.?

33. How much stock must be bought at 88⅓ per cent. in order that by selling out when the stocks are at 90¼, twenty guineas may be gained?

34. Find the value of ³⁄₇ of 1⅖ of 12½, and divide the result by 13¾.

35. Find the value of $\cdot 2\dot{7}$ of £1, and reduce 13 lbs. 15 oz. to the decimal of a cwt.

36. What sum of money laid out in the 3 per cents. at $89\frac{1}{8}$ will give an income of £50?

37. Add together 3·14159, 16·321, ·02, 4·59241; and divide the sum by 7500.

38. The weight of a cubic foot of water is 1000 ounces; find the weight of water necessary to fill a cistern whose length is 4 ft. 6 in., breadth 3 ft., and depth 4 ft.

39. Find the difference between the amounts at simple and compound interest, of £895 16*s.* for 2 years at $3\frac{1}{2}$ per cent.

40. What will be the expense of boarding the floor of a room, at 2*s.* $1\frac{1}{2}$*d.* per square yard, the length of the room being 21 ft. 10 in. and the breadth 18 ft. 4 in.?

41. What is the expense of a diagonal drain in a field 100 yds. long, and 80 yds. broad, at 1*s.* 3*d.* per yd.

42. At what rate per cent. will £625 amount to £765 12*s.* 6*d.* in 5 years, simple interest?

43. What fraction of 3 half-crowns is $3\frac{1}{3}$*s.*; and what decimal of £1 is $\frac{3}{7}$ of 2 guineas?

44. A person earns £175 a year, and pays an income-tax of 14*d.* in the £; what is his net income?

45. If 4 acres of land will maintain 6 horses for 8 weeks, for how long will 10 acres maintain 12 horses?

46. If 3 lbs. of tea be worth 8 lbs. of coffee, and 3 lbs. of coffee be worth 10 lbs. of sugar, how many lbs. of sugar can be got for 15 lbs. of tea?

47. If I lose $1\frac{1}{2}$*d.* in 3*s.* 4*d.* how much is that per cent.?

48. Find the true and ordinary discount of £85 for 16 months at 5 per cent.

49. If 14 horses eat 56 bushels of oats in 16 days, how many days' consumption would there be in 120 bushels for 20 horses?

50. A person, who was owner of $\frac{2}{5}$ of a copper mine, sold $\frac{3}{4}$ of his share for £1800; what was the value of the whole mine?

51. At what rate per cent. will £956 amount to £1314 10*s.* in $7\frac{1}{2}$ years, simple interest?

52. If $\frac{2}{3}$ oz. (Avoirdupois) of an article cost $\frac{1}{5}$ of a shilling, what will $\frac{5}{8}$ lb. of the same cost?

53. If 850 men are besieged in a fortress, and have provisions only for 3 months, how many men must leave the fortress, so that the rest may have provisions for 5 months?

54. A gentleman bequeaths £12,000 to his three sons *A*, *B*, and *C*: to *A* a sum unknown, to *B* twice as much as to *A*, and to *C* as much as to *A* and to *B*. What had each son?

55. If 8 cannon in 1 day expend 48 barrels of powder, in how many days will 24 cannon expend 3168 barrels?

56. If the carriage of 5 cwt. 3 qrs. for 150 miles cost £3 9*s*., what must be paid for the carriage of 7 cwt. 2 qrs. 25 lbs. for 64 miles at the same rate?

57. In what time will £547 15*s*. amount to £629 18*s*. 3*d*. at 5 per cent. per annum?

58. What principal put out to simple interest for 5 years will amount to £570 16*s*. 6*d*. at 4 per cent. per annum?

59. Find the true discount and present worth of £187 5*s*. 6*d*. due at the end of 3 yrs. at $4\frac{1}{2}$ per cent.

60. Find the smallest number that can be divided by 8, 12, 18, 24, 27, without leaving any remainders.

61. What is the least number which taken from 387564 leaves a remainder divisible by 70; and what is the least number which added to 387564 gives a sum divisible by 65?

62. Simplify the following expressions:

$$37\frac{1}{2}-\frac{3}{4} \text{ of } 15; \text{ and } \left(37\frac{1}{2}-\frac{3}{4}\right) \text{ of } 15.$$

63. In 40 guineas how many sevenths of £1, and how many fifteenths of 1 shilling?

64. What fraction added to the sum of $1\frac{2}{3}$, $\frac{3}{4}$ of $1\frac{5}{8}$, and $7\frac{1}{2}$, will amount to $17\frac{5}{8}$?

65. State what is necessary in comparing the magnitude of different fractions; and hence find the comparative value of $\frac{11}{12}$, $\frac{15}{16}$, $\frac{20}{21}$, $\frac{24}{25}$.

66. Arrange in order of magnitude, beginning with the lowest, the fractions $\frac{3}{4}$, $\frac{4}{5}$, $\frac{3+4}{4+5}$.

67. Shew that the simple fraction which expresses the value of $\frac{1}{3}$ of $\frac{\frac{2}{7}}{\frac{1}{14}}$ of $\frac{\frac{2}{3}}{1+\frac{2}{3}}+\frac{2}{7}$ of $\frac{2}{5}$ is equal to $\frac{1}{5}$ of $\frac{2}{7}$ of $\frac{1}{3}$ of $30\frac{1}{3}$.

68. What fraction having 12 for its den^r is equal to $\frac{5\frac{1}{3}}{16}$?

69. Shew, without assuming any *Rule*, that the area of the rectangle whose adjacent sides are $5\frac{1}{6}$ ft. and $3\frac{3}{4}$ ft. $=19\frac{9}{24}$ sq. ft.

70. Shew by a diagram that $30\frac{1}{4}$ sq. yds. $=1$ sq. pole.

71. Shew also by diagrams that $(\frac{1}{2}$ ft.$)^2=\frac{1}{4}$ sq. ft., and that $\frac{1}{5}$ ft. × $\frac{1}{6}$ ft. $=\frac{1}{30}$ sq. ft.

72. Shew by a diagram that there are 640 acres in a square mile.

73. From a square containing 1 acre there are subtracted 16 rectangular plots, each 25·2 yds. long, and 10·5 yds. broad; how much is left?

74. Assuming that $(\sqrt{5}-\sqrt{3})\times(\sqrt{5}+\sqrt{3})=5-3$, and similarly of any other pairs of numbers, find the value to three places of decimals of the quantities $\frac{1}{\sqrt{5}-\sqrt{3}}$, $\frac{\sqrt{3}}{\sqrt{5}-1}$, and $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

75. I pay away $\frac{1}{6}$ of my money, and $\frac{3}{5}$ of the remainder; then $\frac{1}{4}$ of what remains, and lastly, $\frac{1}{16}$ of the original sum; what were the fractional parts remaining after the 3rd and the last payments?

76. Divide £1500 in the ratio of 7, 10, and 13.

77. If 35 bushels of oats are eaten in 17 days by 15 horses, how many days will $115\frac{1}{3}$ bushels last 102 horses?

78. How many cubes, whose edges are 15 inches, could be filled from a full cistern $4\frac{1}{2}$ ft. long, $3\frac{1}{3}$ ft. wide, and $2\frac{1}{2}$ ft. deep?

79. *A* and *B* do a piece of work in 7 days, which *A* alone could accomplish in $35\frac{1}{3}$ days; in what time would *B* do it by himself?

80. What must be the poundage so that a person whose rent is £130 shall pay a rate of £8 2*s*. 6*d*.?

81. What is the superficies of a cube whose edge is $3\frac{1}{3}$ inches?

82. Find the volume of the same cube.

83. What number multiplied by $\frac{3}{5}$ of $\frac{1}{4}$ of $7\frac{1}{3}$ will give as a result $3\frac{11}{15}$?

84. The product of two numbers is 40, and the square of the less is 1296; required the numbers.

85. What would 5*s*. 10*d*. per day come to in a year?

86. What is the price of 4108 articles at 2*s*. $11\frac{1}{4}$*d*.?

87. Find the cost of 1003 articles at 19*s*. $11\frac{3}{4}$*d*.?

88. At what rate per cent. will £165 gain £22 5*s*. 6*d*. in 3 years?

89. In what time will £55 12*s*. 6*d*. amount to £65 12*s*. 9*d*. at 3 per cent. per annum?

90. Required the rate per cent. at which £250 will amount to £267 3*s*. 9*d*. in $2\frac{1}{2}$ years, Simple Interest.

91. What is the simple interest of £265 10*s*. for 5 years 15 weeks, at 3 per cent., reckoning 52 weeks a year?

92. If linen be bought at 1*s*. 5*d*. a Flemish ell, and sold at 2*s*. 4*d*. a yard, what is the gain per cent.?

93. A house was bought for £550, and sold for £570; what was the gain per cent.?

94. A grazier bought 8 oxen for £97, in addition to which they cost him in keep $7\frac{1}{2}$ per cent. of this money. He sells half of them for

£15 10*s.* each; at what price each must he sell the rest, to gain 15 per cent. on his entire outlay?

95. If 1*s.* gain 2½*d.* what is the gain per cent.?

96. Find the value of 8888 articles at £1 0*s.* 8*d.*

97. What is the worth of 3303 articles at £1 19*s.* 7½*d.*?

98. Find the discount of £415, due 9 months hence, at 5 per cent. Simple Interest?

99. What is the present worth of £1281 5*s.*, due 6 months hence, at 5 per cent. per annum?

100. What sum will amount to £306 5*s.* in 9 years, at 2½ per cent. Simple Interest?

101. Find the value of 5555 articles at 1*s.* 6¼*d.*

102. A man in walking takes ·88 yds. each step, and is 5 hrs. 24 min. in going a journey of 12·15 miles; required the number of times he steps in a minute.

103. If an ounce of tea cost ·175 of 1*s.* 8*d.*, how much can be bought for 8 guineas?

104. From ·725 crowns take 2·25 shillings, and reduce the remainder to the decimal of half-a-crown.

105. What fraction of an English ell is $\frac{4}{9}$ of an inch?

106. From $22\frac{2}{3}$ take $11\frac{1}{3}$; and multiply the remainder by 5 times $7\frac{1}{2}$.

107. If 2 cwt. of coal cost 1*s.* $11\frac{1}{3}$*d.*, what will $12\frac{3}{8}$ tons cost?

108. A person in going a journey of 40·3 miles performs the first 14 miles in 1·7 hours; at what rate must he go the other part, so as to complete the whole journey in 4·2 hours?

109. Which is the greatest, and which the least of the following fractions, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{5}{8}$, $\frac{7}{9}$, $\frac{8}{11}$?

110. Find the value of $16\frac{3}{4}+2\frac{7}{24}+22\frac{7}{30}+17\frac{5}{6}+3\frac{1}{5}$.

111. Divide the difference between $1\frac{3}{5}$ of $5\frac{1}{2}$ and $1\frac{2}{3}$ of $6\frac{2}{3}$ by the product of $\frac{3}{4}$ and 9.

112. What fraction of a French ell is $\frac{5}{9}$ of an English ell?

113. Divide $15\frac{3}{8}$ equally amongst 9 persons.

114. Reduce $\frac{15}{16}$ to a complex fraction whose denominator is $39\frac{1}{3}$.

115. How many twelfths are there in $29\frac{1}{3}$?

116. The income-tax is 15*d.* in the pound; what is that per cent.?

117. By selling coats at 19*s.* 3*d.* each, 5 per cent. is lost; what should they be sold at, so as to gain 10 per cent.?

118. If ribbon be bought for 2¼*d.* per yard, and sold at 5¼*d.*, what is the gain per cent.?

119. If a ship be bought for £7381 5*s.* and sold for £8797 10*s.*, what was the gain per cent.?

120. What is the value of £3750 stocks in the 3 per cents., at 87⅝?

121. How much money must be invested in the 4½ per cents. at 99⅜, so as to yield an annual income of £360?

122. What amount of 3¼ per cent. Stock would yield £81 5*s.* per annum?

123. Find what decimal multiplied by 72 will express the sum of $\frac{1}{8}$, $\frac{7}{18}$, and 4¼.

124. Find the value of 4107 articles at £9 9*s.* 9*d.*

125. What is the exact discount on £5 12*s.* 9*d.*, due 12 months hence, at 2½ per cent.?

126. What is the difference between the interest and discount on £75 18*s.* 9*d.* for 3 months at 5 per cent.?

127. What is the gain per cent. by buying eggs at 15 for a shilling, and selling them at a penny each?

128. What should an article which cost 8*s.* 4*d.* be sold for, so as to gain 15½ per cent.?

129. Find the cost of 4107 articles at £1 0*s.* 11½*d.* each.

130. A cow was bought for £10 15*s.* and sold at a loss of 7½ per cent.; what did she fetch?

131. If 1*s.* gain 1⅓*d.*, what is the gain per cent.?

132. A court-yard is 30 ft. 10 in. long, and 12 ft. 11 in. broad; what will the paving come to at 16*d.* per sq. ft.?

133. Of what amount must I take $\frac{11}{12}$ to produce a guinea?

134. In 1851 the population of a town was 400,000; supposing at every decennial census it be found to have increased 20 per cent. on the preceding one, what will be the population in the year 1901?

135. A man builds 12 cottages at a cost of £925, and lets them at £5 10*s.* per annum. One on an average is always vacant; and of the rents that are due he loses 7½ per cent. If the yearly repairs cost ⅜ per cent. of the outlay, what rate of interest does he obtain for his money?

136. What is the commission on £4695 at 5*s.* 9*d.* per cent.?

137. What is the amount of ·0125 per cent. of a population of a million?

138. The duty on sales by auction was once 5 per cent. on the purchase-money, and an additional 5 per cent. on this duty; what was the total cost of an article knocked down at £51?

139. If 5*s*. 3*d*. gain 2*s*. 1½*d*., what is the gain per cent.?

140. Find the cost of repairing 7 m. 7 fur. 35 p. of a road, at £3 17*s*. 4½*d*. per mile.

141. Of what fraction must I take ⅗ to receive $\frac{11}{12}$?

142. A banker receives in deposits £16250 a year, for which he pays 3½ per cent. yearly, and he lets it out at 5 per cent. payable half-yearly; what will be his annual profit?

143. The price of ·0625 lbs. of coffee being ·458$\dot{3}$ shillings, what is the cost of ·075 of a ton?

144. How many bricks will be required for a wall 25 yds. long, 15 ft. high, and 1 ft. 10½ in. thick, each brick being 9 in. long, 4½ in. wide, and 3 in. thick?

145. If the papering of a room 28 yds. round and 4 yds. high cost £5 10*s*., what will be the cost of similarly papering another room, 24 yds. round and 3½ yds. high?

146. Find the equated time of payment of a debt, of which ⅓ is due in 4 months, ⅕ in 7 months, and the rest in 8 months.

147. Find the side of a square area, the paving of which at 4*s*. 2*d*. per sq. yd. cost £56 0*s*. 10*d*.

148. What must be paid for painting the outside of a box 5½ ft. long, 3¾ ft. wide, and 2⅝ ft. deep, at 1*s*. 8*d*. per sq. yard?

149. Extract the cube roots of 59319, and $4\frac{37}{5832}$.

150. A square field contains 15 a. 2 r. 20 p.; find its side in chains.

Exs. 67.

MISCELLANEOUS EXAMPLES.

1. What will be the net value of a legacy of £333 6*s.* 8*d.*, after paying duty at the rate of 3 per cent.?

2. Two clocks point to 2 at the same instant; one loses 7 seconds, and the other gains 8 seconds in 24 hours; when will one be half an hour before the other, and what time will each clock then shew?

3. A bankrupt owes 3 creditors £10,000, 10,000 guineas, and 10,000 shillings respectively; but his property is worth only £7000: find how much in the pound he will pay, and how much each creditor will receive?

4. If 100 articles are bought at 3 a penny, and 100 more at 2 a penny; at what price must they be retailed so as to gain 25 per cent.?

5. A vessel containing 384 gallons is emptied by 3 taps; the first and second together empty it in 32 minutes; the first and third in 24 minutes; and the second and third in 16 minutes; how many gallons will each tap discharge in a minute?

6. How many square feet of board will be required to make a rectangular box, of which the length, breadth, and depth are 3¼ ft., 2¼ ft., and 1 ft. 2¼ in. respectively?

7. If a room be 32 ft. long, 26 ft. wide, and 14 ft. high, what will be the expense of papering it at 3*s.* per square yard, allowance being made for three windows, each 10 ft. by 6¼ ft., and two fire-places, each 9½ ft. by 6 ft.?

8. What sum of money must be paid down, in order to receive £360 10*s.*, 2 yrs. hence, allowing 3½ per cent. Compound Interest?

9. Find the simple fraction which expresses the value of $7+\sqrt{6\frac{1}{4}}$, when divided by $6\frac{1}{3}$ times $(3+\sqrt[3]{3\frac{3}{8}})$.

10. How many cubes, whose edges are ¾ in. long, can be contained in a box, of which the base is 18 sq. inches, and height 7½ inches?

11. The volumes of spheres are in proportion to the cubes of their radii: the radii of two spheres are in the ratio of 4 to 5; and the weights of equal portions of the smaller and larger spheres are as 12 : 7; given that the weight of the smaller sphere is 256 lbs., find the weight of the larger one.

12. At what time between one and two o'clock will the hour and minute hands of a watch make an angle of 60 degrees with each other?

13. If a pound Troy of English standard gold $\frac{11}{12}$ fine be worth £46 12*s.* 6*d.*, what is the value of a coin weighing 7 dwts. 11 grs., in which 924 parts in 1000 are pure gold?

14. The wheels of a cart are $2\frac{1}{2}$ yards asunder, and the inner wheel describes the circumference of a circle of radius 20 yds.: find the difference of the paths of the wheels, having it given that the circumference of a circle = 3·1416 times its diameter.

15. Two men are walking in the same direction, the distance between them at starting being 100 yds.; the first walks 45, and the second 49 yds. in 50 steps; how many steps will have been taken when they are together?

16. How much paper, $\frac{3}{4}$ yd. wide, will be sufficient to paper a room 22 feet 5 inches long, 12 feet 1 inch broad, and 11 feet 3 inches high? and how much will it cost at $4\frac{1}{2}$*d.* per yard?

17. A river 30 feet deep, and 200 yds. wide, is flowing at the rate of 4 miles an hour; find how many cubic feet of water run into the sea per minute; also the number of tons, supposing a cubic foot of water to weigh 1000 ounces.

18. A clock gains $3\frac{1}{2}$ minutes per day; how should its hands be placed at noon, that it may point out the true time at $7\frac{1}{2}$ in the evening?

19. A person performs $\frac{2}{7}$ths of a piece of work in 13 days; he then receives the assistance of another person, and the two together finish it in 6 days; in what time could each do the whole work by himself?

20. A tradesman buys goods for £189 15*s.* 6*d.*, and sells them in 6 months for £253 0*s.* 8*d.* ready money: how much is that per cent. per annum profit?

21. If the value of £1 sterling varies from 25·15 francs to 26·75 francs, what is the variation in value of 100 guineas?

22. A person receives his rent, and after paying an income-tax of 7*d.* in the pound has £553 7*s.* 6*d.* left; what did he receive?

23. A field of $7\frac{1}{2}$ acres is planted with potatoes in rows; the distance between each row is 15 inches; how many yards of potatoes are there in the field?

24. Three masons, *A*, *B*, *C*, are to build a wall; *A* and *B* could together build it in 12 days, *B* and *C* in 20, and *A* and *C* in 15; in what time will they build it when they all work together?

25. If 3 miles, 4 furlongs, 93 yards be run in 6 minutes 4 seconds, how much is that short of the rate of a mile per minute?

26. Multiply $\dfrac{2\frac{3}{8}}{4\frac{3}{4}}$ by $\dfrac{2\frac{1}{52}}{13\frac{6}{13}}$, and divide $\sqrt{2\frac{1}{4}}$ by $\sqrt[3]{3\frac{3}{8}}$.

27. The comparative weights of coal and water are as 1·12 and 1; also a cubic foot of water weighs 1000 oz.; find the edge of a cubical block of coal which weighs 2000 tons.

28. The discount on £500 due 4 years hence is 250 marks; find the rate of interest.

29. Shew what factor is wanting in the number 32 to make it a perfect cube.

30. Required the least number which multiplied by 64 will make it a perfect 5th power.

31. *A* sets out from Cambridge to London ($51\frac{1}{4}$ miles) at the rate of 8 miles an hour, and *B* sets out at the same time from London to Cambridge at the rate of $9\frac{1}{2}$ miles an hour; at what distance from each place will they meet?

32. A tradesman marks his goods with two prices; one for ready money, and the other for credit of 6 months: what fixed proportion ought the two prices to bear to each other, allowing 5 per cent. Simple Interest?

33. What must be the least multiplier of the number 225, so that the product may be a perfect cube?

34. If the ratio of the diameter of a circle to its circumference be 113 : 355; and if the length of $\frac{1}{360}$th part of the earth's circumference be $69\frac{1}{12}$ miles, what is the earth's diameter?

35. How much per cent. is 14*s.* 6*d.* of £3 10*s.*?

36. *A* and *B* together can do a piece of work in 30 days; *B* by himself can perform the same in 70 days: in what time could *A* finish it by himself; and how much more of the work does *A* do than *B*?

37. The sun's longitude is increased by 360 degrees in 365 d. 5 h. 48 m.: what is his average daily motion?

38. Simplify the following expression, leaving one surd in the numerator of the resulting fraction:—

$$\sqrt{6^2-5^2}\times\frac{5}{\sqrt{6^2+5^2}}+\sqrt{6^2+5^2}\times\frac{5}{\sqrt{6^2-5^2}}.$$

39. *A* and *B* can do a piece of work together in 30 days; *A* does $\frac{1}{4}$ more than *B*; in what time can they do it separately?

40. Find the square root of the sum of the squares of ·2, ·4, ·6, ·86.

41. If an acre of land be bought for 6*d.* per foot, at what price per yard must it be sold to gain £136 2*s.* 6*d.*?

42. From a rectangular plank 1 foot broad and 2 inches thick, what length must be cut off to be worth 25*s.*, the value of the timber being 5*s.* per cubic foot?

43. Simplify the following expressions:—

$$1+\cfrac{2}{3+\cfrac{1}{4+\cfrac{1}{5\frac{2}{3}}}}. \qquad \frac{\sqrt{1+\frac{1}{3}} \div \sqrt{1-\frac{1}{5}}}{\sqrt{1+\frac{1}{3}} \times \sqrt{1-\frac{1}{5}}}.$$

44. Find the value of $3 \div \sqrt{10}$ to four places of decimals.

45. If the volume of a cylinder be obtained from this product, height × area of the base; find the area of the base of a cylinder whereof the volume is 1 cubic foot, and the height $7\frac{1}{2}$ inches.

46. Compare the volumes of two cylinders whose bases are in the ratio of 3 : 4, and altitudes as 5 : 6.

47. A person rents a piece of land for £120 a year. He lays out £625 in buying 50 bullocks. At the end of the year he sells them, having expended £12 10*s.* in labour. How much per head must he gain by them, in order to realize his rent and expenses, and 10 per cent. on his original outlay?

48. A cubical box contains 37 solid feet and 64 solid inches: find (1) the number of linear inches in the edge; (2) in the diagonal of each face; and (3) in the diagonal of the box.—(*See Appendix, Note to Art. Ratio.*)

49. A person finds that his net income, after deducting the income-tax of 7*d.* in the pound, is £233; find the amount of income.

50. A pile of cannon shot has a rectangular base, the sides of which contain 7 and 6 shot respectively; find the number of shot in the whole pile.

51. What is the unit of measurement when a mile is 160 units? Determine that common unit which will express 12960 minutes and 20160 minutes in the smallest possible integral numbers.

52. The areas of circles are proportional to the squares of their radii; find the ratio of the areas of two circles which have 2 feet and $\frac{1}{2}$ an inch as the values of their respective radii.

53. Find the ratio of the diameters of two circles, such that the area of one may $= 5\frac{4}{9}$ times that of the other.

54. Two cubical boxes have edges respectively $\frac{3}{4}$ inch, and 2 ft. 3 in.; find the ratio (1) of their surfaces, and (2) of their volumes.

55. A clock has its face marked so as to shew 24 hours in a day; and on a certain evening half an hour after sunset it was set at 24 o'clock. The morning following it was 8 min. past 4 by a common clock when it was 4 minutes past 8 by this clock. Find the time of sunset the previous evening.

56. Given that the ratio of the circumference of a circle to the diameter $= 3\cdot1416 : 1$; also that the length of an arc of a circle opposite to any angle at the centre is proportional to the degrees, &c. in that angle: find the length of the radius of that circle, in which an arc of 4000 miles is opposite to an angle containing 8·58 seconds.

57. The space through which a body falls in vacuo near the surface of the earth is proportional to the square of the number of seconds in the time of falling: if a body fall through 16·1 feet in the first second, find how far it will fall in 8 seconds, and in the ninth.

58. The time of oscillation of the pendulum of a clock $= 3\cdot1416 \times \sqrt{\frac{39\cdot2}{32\cdot2}}$ seconds, where 39·2 inches is the length of the seconds pendulum; find the alteration in the time of oscillation when the pendulum is lengthened $\frac{1}{10}$ of an inch. Find also how many seconds the clock will gain or lose in 24 hours.

59. Taking the same value as in the last question for the time of oscillation, where 32·2 is the measure of the force of the earth's attraction, find the alteration of that measure, if the length of the pendulum be increased by $\frac{1}{1000}$th part, and the time of oscillation be the same.

60. The difference between the year in the Julian calendar and the true year is ·007736 days; the Gregorian calendar corrects by omitting 3 days in 400 years: find how much error would have accumulated under that calendar, from A.D. 325 to A.D. 1848, and how soon the error will amount to a day.

61. In a block of wood, a hole is made 12 in. long and 1 sq. inch in section: the largest possible cylinder is placed in the hole; how much is unoccupied?—(*See question* 45.)

62. A cubical box, 1 inch high, is filled with water; 8 equal spheres of $\frac{1}{2}$ in. diam. are placed in it; what volume of water will remain,—having it given that the volume of a sphere $= \frac{4}{3} \times 3\cdot1416 \times$ the cube of the radius.

63. Will $\frac{21}{140}$ produce a circulating decimal?

64. A cube has an edge 2 ft. 6 in. long; find the ratio between the sum of the areas of the semicircles described on its edges, and the whole surface of the cube, having it given that the area of a circle $= 3\cdot1416 \times$ the square of the radius.

65. A wall is 15 ft. 8 in. long, and 11 ft. 6 in. broad, and has in it a door-way 6 ft. 3 in. by 2 ft. 4 in.; find the number of bricks of $165\frac{7}{12}$ solid inches contained in it, when the thickness is 11 inches.

66. If a sovereign weigh 5 dwts. $3\frac{1}{4}$ grs., 1 part out of 12 being copper, and the rest pure gold; find what fraction of a cubic inch the gold constitutes, having given that a cubic inch of water weighs 252·458 grains, and that gold is 19·362 times as heavy as water.

67. Reduce the following Arithmetical expression to its simplest form:

$$\left\{\left(\frac{9}{10}+2\tfrac{1}{2}\right)-(2\tfrac{1}{5}-1\tfrac{3}{4})\right\}\times\{(5\tfrac{1}{4}+7\tfrac{2}{3})\div 16\tfrac{1}{10}\}.$$

68. A cistern has 3 pipes, *A*, *B*, and *C*; *A* and *B* can fill it in 3 and 4 hours respectively; and *C* can empty it in 1 hour; if these pipes be opened in order at 1, 2, and 3 o'clock, find when the cistern will be empty.

69. A person sells out of the $3\frac{1}{2}$ per cents at $93\frac{1}{2}$ as much stock as produces £9350; at what price must the 4 per cents be, so that the above sum when invested in them shall produce an increase of £10 income?

70. If $\frac{1}{5}$ of a sheep be worth £$\frac{3}{4}$, and $\frac{4}{7}$ of a sheep be worth $\frac{1}{12}$ of an ox, what sum must be given for 50 oxen?

71. A garden walk, 4 ft. wide, is carried round a circular plot 3 yds. in diameter; find the price of gravelling the walk per foot, if the whole cost be 6*s*.

BOOK-KEEPING.

THE object of Book-keeping is to enable a merchant to ascertain

(1) How he stands with respect to all the persons with whom he has any money dealings.

(2) The money value of all that he possesses, and all that he owes, which are technically called his *Assets* and *Liabilities*. If the Assets exceed the Liabilities, he is said to be *solvent;* otherwise he is termed *insolvent*.

(3) To ascertain how, if he is insolvent, he has come to be so; and if not, in what manner his profits have been realized, so as to render him solvent, after all trade and domestic expenses have been paid.

This third account is termed the Profit and Loss account.

To ascertain (1), the merchant must have accounts open under the name of every *person* with whom he deals; and these are called *personal* accounts: and if he keeps his accounts only in this simple form, he can, by merely balancing these personal accounts, and taking an inventory of stock and cash, tell at any given time how he stands. And this system of Book-keeping is termed Single Entry.

But this will not tell him how he came into his present position; or shew upon what goods he has made or lost money. But if he produces (2), i.e. accounts of *things* bought or sold, paid or received, in addition to the accounts of *persons*, then he is enabled to tell how he stands with reference either to his original capital, both in goods and money, or to the state of that capital at any given time.

These accounts, being headed by the names of things and not of persons, are called *Nominal* Accounts. And it is the keeping of these, in addition to the Personal Accounts, which constitutes Double Entry.

The books required for book-keeping are (1) an Invoice-book; (2) a Day-book, or Waste-book; (3) a Cash-book; (4) a Ledger; (5) a Bill-book. This last is seldom used where the transactions are small. A memorandum of any transaction recorded in any one of these books is termed an Entry. A specimen of each of the above books will be given.

By Single Entry, every transaction comes direct from the books already mentioned into the Ledger; in each book the page of the Ledger is marked where the entry may be found, and in the Ledger the page or folio is also marked of the book from which it is brought. But in Double Entry there is generally used an additional book called the Journal, which is an intermediate book between those mentioned and the Ledger, so that nothing passes into the Ledger except through this book.

The transferring of the entries from these various books into the Ledger is termed *posting.*

Before proceeding to give specimens of these books, it will be advisable to call attention to a few preliminary principles, as to the mode of keeping accounts.

All books are arranged with two sides, a *debtor* to the *left,* and a *creditor* to the *right.* These are briefly written Dr. and Cr.

Under any personal or nominal account, whatever is paid away, or parted with, by the person or name heading the account, is put to the Cr. side. And whatever has been received by the same, and which must therefore be accounted for, is Dr. In fact, we may say briefly, whatever is *in,* is Cr.; whatever is *out,* is Dr. When any item is thus put to the *credit* side of an account, the entry is preceded by the word "By." And when it is put to the debtor side of an account, it is preceded by the word "To."

Hence, if an article, as tea, is sold to a person on credit, there is an entry under his personal account, "Dr. To Tea;" and when he pays for it in cash, or bill (of which more will be said presently), there is an entry under the same account to his credit, "Cr. By Cash," or "By Bill;" and the two sides of the account balance.

But when the above goods, as tea, were sold to him, while he was the receiver, the article or articles delivered were payers of the goods; and under the nominal account of "Tea," there would be an entry "Cr. By Tea to A. B." And when this is paid, the nominal account headed "Cash" would have put under it the entry "Dr. To Cash from A. B." So that we have this arrangement; a Dr. entry to a personal account is balanced by a Cr. entry to a nominal account for goods: and a Cr. entry to a personal account for cash received is balanced by a Dr. entry to the nominal amount headed Cash.

Again, let the firm be buyers of tea from any merchant, then while under his personal account there is an entry "Cr. By Tea," there will be a corresponding Dr. entry under the nominal account headed "Tea." And when the tea is paid for by the firm, the merchant will be Cr. by the amount, and cash will be the Dr., since cash has received it.

Hence it is evident that if the posting be correct, all the entries on the Dr. side in the Ledger must exactly balance all the entries on the Cr. side. And this comparison of the totals of the two sets of Dr. and Cr. entries, for the purpose of testing the accuracy of the posting, is called a "Trial Balance," and is simply a trial, as to whether the entries have been correctly posted from the Journal, or the Day-book if journalized, into the Ledger.

All the above-mentioned Nominal accounts might have been classed under one head, viz. that of the firm, who manifestly are Drs. or Crs., according as each of their customers is Cr. or Dr.; and so a balance might be struck as before. But by the above classification, or division into several nominal accounts, each embracing but one class of goods, we are enabled to ascertain, not only *what* profits have been realized, but *how* they have been realized.

We will now describe the nature and use of

BILLS OF EXCHANGE.

A Bill is a promise to pay, usually at some bank in London, a certain sum mentioned on the bill, at the end of a time agreed upon, say 2, 3, or 6 months, and is drawn up on stamped paper in the following form.

LIVERPOOL, *Oct.* 27, 1860.

£56. 15*s*. 0*d*.

Three months after date pay to my order the sum of Fifty-six Pounds Fifteen Shillings for value received.

A. B.

To C. D. & Co.
Chesterfield.

Accepted payable at
R. T. & Co. Bankers.
London.
C. D.

Here C. D. owes the money to A. B. and by writing across the bill his name and that of the banker at whose bank it is made payable, he is said to *accept* the bill, and thus become responsible for its payment. A. B. is called the *drawer*, and C. D. the *acceptor*. A. B. who receives this bill writes his name on the back of it, if he parts with it, thereby making himself responsible to the person to whom he pays the bill, and he is then called the *indorser;* the person to whom he pays it, if his name is specially mentioned on the back of the bill, is called the first *indorsee*, and if he pays it to a third party, *he* must *indorse* it, i.e. write *his* name on the back also, and thus render himself responsible for the payment, if C. D. and A. B. fail to pay, or, as it is commonly termed, to *meet* it.

In any general statement of the business of C. D., commonly called a Balance Sheet, of which more will be said hereafter, all such bills which were accepted by C. D. and still unpaid or running, would be entered on the *debtor* side as "Bills payable." If in the course of his business he had received bills accepted by other parties, these would (subject to discount) be considered as cash, provided the acceptors were considered able to meet them: and they would be entered to his *credit* as "Bills receivable."

But such bills might also appear on the debtor side of his Balance Sheet, because he might have had some passing through his hands, accepted by parties that had since failed; and since he has indorsed them, he would be wholly, or in part, liable to be called upon for them. Hence in his Balance Sheet bills receivable in hand would be on his Cr. side, and bills receivable parted with, and not yet at maturity, would be on the Dr. side, so far as the acceptors are unable to meet them.

THE INVOICE-BOOK

This is used merely to record the invoices of all goods ordered into stock.

Specimen.

Chesterfield, *October* 27, 1860.

A. B.

Bought of C. D. and Co.

	£	*s.*	*d.*
4 Chests of Tea, each 56 lbs. at 3*s.* 4*d.*	37	6	8
3 cwt. 3 qrs. of Sugar, at 56*s.* per cwt.	10	10	0
	47	16	8

The above Invoice-book belongs to A. B., who is the buyer of the goods.

The more usual plan of keeping the Invoice-book is by pasting all Invoices received, in a book prepared for the purpose, according to their dates.

THE DAY-BOOK.

A Day-book, in its most limited sense, is one in which the transactions of each day are recorded. But sometimes it is made to contain a record of all the transactions of each day, whether of sales or purchases; bills accepted, or bills taken in payment; cash received or paid. And these various items from it have to be transferred to the Journal, the

Bill-book, the Ledger, and Cash-book respectively. If no Journal is kept, the Day-book may be made a record of all transactions both Dr. and Cr. tnat occur in the business, and it may be then said to be "journalized.'

Sometimes this book is called a Waste-book, especially when it embraces transactions as enumerated above. But if kept merely as an ordinary Day-book, it gives accounts only of sales and present payment. And in a very simple business, where but few articles are dealt in, if the amount of these sales be carried forward into a separate column at the end of each day, and then at the end of each month and year, the results for each year may be entered in a spare page of the Ledger, and thus give a view of the progress of the business.

F.L. stands for "Folio in Ledger," C.B. for "Folio in Cash-Book."

DAY-BOOK.

F. L.		C. B.	Bought.	Sold.	Miscellaneous.	Total.
	January 1, 1861.					
8	Bought of Phillips and Co.					
12	10 Chests of Tea (1050 lbs.) at 1/8		87 10 0			
13	8 pieces of Linen (128 yds.) at 33/6		13 8 0			100 18 0
1	Lodged in Joint Stock Bank				500 0 0	500 0 0
	2					
2	Sold J. Smith and Co.					
10	4 Bales of Cotton at £25 per			100 0 0		100 0 0
13	7 Pieces of Linen at 35/6 per			12 8 6		12 8 6
	4					
3	Bought of T. Martin					
	Shop fittings				45 10 0	45 10 0
	8					
2	Received from J. Smith & Co. in					
	payment of their acct.; Cash				40 0 0	
	Bill at 2 mo. (4 Jan.—7 Mar.) No. 1.				72 8 6	112 8 6
	10					
4	Bought of B. Wilson					
	Gas fittings				17 15 0	17 15 0
	15					
5	Bought of Wm. Hinds					
10	10 Bales of Cotton at £20 per		200 0 0			
11	5 Hhds. of Port Wine at £16/5 p.		81 5 0			281 5 0
	Carried forward		382 3 0	112 8 6	675 18 6	1170 5 0

DAY-BOOK.

F. L.		C. B.	Bought.	Sold.	Miscellaneous.	Total.
	Amount brought forward		382 3 0	112 8 6	675 13 6	1170 5 0
	January 16					
3	Accepted T. Martin's Bill on me. No. 1, at 2mo. (16 Jan.—19 Mar.)				45 10 0	45 10 0
	18					
6	Sold Atkins and Co.					
11	4 Hhds. of Port Wine at £17/10 p.			70 0 0		70 0 0
6	Received from Atkins & Co. No. 2. Their Acceptance at 1 mo. (Jan. 11—14 Feb.)				70 0 0	70 0 0
	28					
1	Lodged in Joint Stock Bank				40 0 0	40 0 0
	February 6					
5	Accepted Wm. Hinds' Bill on me at 1 mo. (Jan. 24—27 Feb.) No. 2.				200 0 0	200 0 0
	14					
6	Received payment of Bill, No. 2. Accepted by Atkins and Co.				70 0 0	70 0 0
	26					
5	Paid Bill, No. 2. Accepted to Wm. Hinds				200 0 0	200 0 0
1	Drawn from Joint Stock Bank				200 0 0	200 0 0
	March 1					
7	Sold Robert Topham					
10	6 Bales of Cotton at £25			150 0 0		
12	8 Chests of Tea (840 lbs.) at 2/0			84 0 0	17 15 0	234 0 0
4	Paid B. Wilson for Gas fittings, &c.					17 15 0
	7					
2	Received payment of Bill, No. 1. Accepted by J. Smith and Co.				72 8 6	72 8 6
	18					
3	Paid Bill No. 1. Accepted to J. Martin				45 10 0	45 10 0
	29					
1	Drawn from Joint Stock Bank				175 0 0	175 0 0
	Carried forward		382 3 0	416 8 6	1811 17 0	2610 8 6

The Specimen of a Day-book represents what we have called a journalized arrangement, because under the head "Sold" is exhibited at one view the total amount of business done; and this can be carried forward, as described in the last paragraph. Of course the total amount of the first three columns is the same as that of the fourth, headed "Total."

THE CASH-BOOK.

The Cash-book contains simply the amounts received and paid every day. It must therefore according to p. 226 have two sides, a *debtor* side, on which are entered all the sums *received*, and for which therefore the cash must be considered responsible, i.e. a *debtor:* and a creditor side, on which are entered all the sums *paid*, and for which therefore the cash must be considered a *creditor*.

For a specimen of the Cash-book we refer to that portion of the Ledger headed Cash. Fo. 15, which will be found to contain all the entries from the Day-book involving cash or bills, both paid and received. All these payments or receipts have been entered in the Ledger under the names of the parties in connexion with whose transactions they have occurred; but they will be on the reverse sides of the Ledger, because for every sum paid or received by a customer there will have been a corresponding sum received or paid by the Cash-book, or, which is the same thing, by the Cash account in the Ledger.

There may appear also a third and different kind of entry; as, when a merchant has drawn a cheque on his own banker, and paid away either a part or the whole of the money. If he has paid the whole, there will be two equal entries on opposite sides; but if he has paid only part of it, there will be one entry on the Dr. side, of the whole cheque drawn; and one smaller entry on the Cr. side, leaving a corresponding balance in hand at the end of the day's transactions.

When a bill comes back dishonoured, and the drawer has to pay it by a cheque on his banker, he must enter the amount on both sides of his Cash-book as money received from his banker, and at the same time as paid away through the same banker to take up the bill. If the acceptor afterwards pays the money, the amount should be entered on the Dr. side of the drawer's Cash-book; and if paid to the banker again it will appear on the Cr. side, or it may remain as a balance in the Cash-box.

THE LEDGER.

The Ledger is a book containing the state of the account of every customer, with the firm: and its entries are obtained from the Day-book and the Cash-book.

Thus, all records of *sales* which have been entered in the Day-book will be transferred to the *debtor* side of the respective accounts in the Ledger because the parties have *received* goods; and all entries in the Day-book or Cash-book of cash received or bills received in lieu of cash, will be entered on the *creditor* side, because the parties have delivered the same.

It might happen occasionally that cash had been advanced to a debtor of the firm, to enable him to meet an acceptance or otherwise; then of course the item would appear on his debtor side in the Ledger.

LEDGER.

Dr.		Joint Stock Bank		Fo. 1.			Contra		Cr.
1861			F. D.B	£ s. d.	1861			F. D.B	£ s. d.
Jan.	1	To Cash	1	500 0 0	Feb.	26	By Cash	2	200 0 0
,,	28	To do.	1	40 0 0	Mar.	29	,, do.	2	175 0 0
				540 0 0			,, Balance		165 0 0
Mar.	31	To Balance brought down		165 0 0					540 0 0

Dr.		J. Smith & Co.		Fo. 2.			Contra		Cr.
1861					1861				
Jan.	2	To Cotton	1	100 0 0	Jan.	2	By Cash	1	40 0 0
		To Linen	1	12 8 6			,, Bills Receivable	1	72 8 6
				112 8 6					112 8 6

Dr.		J. Martin		Fo. 3.			Contra		Cr.
1861					1861				
Jan.	16	To Bills payable (1)	1	45 10 0	Jan.	4	By Shop fittings	1	45 10 0

Dr.		B. Wilson		Fo. 4.			Contra		Cr.
1861					1861				
Feb.	28	To Cash	2	17 15 0	Jan.	10	By Gas fittings	1	17 15 0

LEDGER.

Dr.		Wm. Hinds		Fo. 5.					Contra		Cr.		
1861			F. D.B.	£	s.	d.	1861			F. D.B.	£	s.	d.
		To Bills payable (2)	1	200	0	0	Jan.	15	By Cotton	1	200	0	0
Mar.	31	,, Balance		81	5	0			,, Port Wine	1	81	5	0
				281	5	0					281	5	0
							Mar.	31	By Balance brought down		81	5	0

Dr.		Atkins & Co.		Fo. 6.					Contra		Cr.		
1861							1861						
Jan.	18	To Port Wine	1	70	0	0	Jan.	18	By Bill Receivable(2)		70	0	0

Dr.		R. Topham		Fo. 7.					Contra		Cr.		
1861							1861						
Mar.	1	To Cotton	2	150	0	0							
		,, Tea	2	84	0	0							
				234	0	0	Mar.	31	By Balance		234	0	0
	31	To Balance		234	0	0							

Dr.		Phillips & Co.		Fo. 8.					Contra		Cr.		
1861							1861						
							Jan.	1	By Tea (10 Chests)	1	87	10	0
									,, Linen (8 pieces)	1	13	8	0
Mar.	31	To Balance		100	18	0					100	18	0
							Mar.	31	By Balance brought down		100	18	0

Dr.		Cotton		Fo. 10.					Contra		Cr.		
1861							1861						
Jan.	15	To W. Hinds (10 B.)	1	200	0	0	Jan.	2	By Smith & Co. (4 B.)	1	100	0	0
Mar.	31	,, Profit and Loss		50	0	0	Mar.	1	,, R. Topham (6 B.)	2	150	0	0
				250	0	0					250	0	0

Dr.		Port Wine		Fo. 11.					Contra		Cr.		
1861							1861						
Jan.	15	To W. Hinds (5 Hhds.)	1	81	5	0	Jan.	18	By Atkins & Co. (4 Hhds.)	1	70	0	0
Mar.	31	Profit and Loss		5	0	0	Mar.	31	,, Balance (1 Hhd. in hand)		16	5	0
				86	5	0					86	5	0

LEDGER.

Dr.		Tea		Fo. 12.			Contra		Cr.
1861			F. D.B.	£ s. d.	1851			F. D.B.	£ s. d.
Jan.	1	To Phillips & Co. (10 Chests)	1	87 10 0	Feb.	28	By R. Topham	2	84 0 0
Mar.	31	,, Profit and Loss		5 5 0	Mar.	3	,, Balance (Two Chests in hand)		8 15 0
				92 15 0					92 15 0

Dr.		Linen		Fo. 13.			Contra		Cr.
1861					1861				
Jan.	1	To Phillips & Co. (8 pieces) 33/6	1	13 8 0	Jan.	2	By Smith & Co.	1	12 8 6
Mar.	31	,, Profit and Loss		14 0	Mar.	31	,, Balance (1 piece in hand)		1 13 6
				14 2 0					14 2 0

Dr		Cash		Fo. 15.			Contra		Cr.
1861					1861				
Jan.	2	To Cash, Smith & Co.	1	40 0 0	Jan.	1	By Cash, J. S. Bank	1	500 0 0
Feb.	14	,, Bills receivable (2)	2	70 0 0		28	,, do.	2	40 0 0
	26	,, J. S. Bank	2	200 0 0	Feb.	26	,, Bills payable (2)	2	200 0 0
Mar.	7	,, Bills receivable(1)	2	72 8 6	Mar.	1	,, B. Wilson	2	17 15 0
	29	,, J. S. Bank	2	175 0 0		18	,, Bills payable (1)	2	45 10 0
	31	To Balance		245 16 6					803 5 0
				803 5 0			By Balance brought down		245 16 6

PROFIT AND LOSS.

To Cotton	50 0 0		
,, Port Wine	5 0 0		
,, Tea	5 5 0		
,, Linen	14 0		
	60 19 0	Balance	60 19 0

In Fol. 1 there is an excess of £165 on the Dr. side over the Cr.; this added to the Cr. side of course makes the two sides agree; and it should then be brought to the Dr. side, as representing the state of the account, or as commencing a fresh account. In Fol. 5 is an example of the position of the balance being reversed.

In the Profit and Loss account, had there been any losses, they would have diminished the balance of £60 19*s*.

THE BILL-BOOK.

The bills payable and receivable account contains all the bill transactions.

(1) The Bills receivable. The Dr. side of this account contains all the bills received from other parties, or drawn upon them; the parties on whom they are drawn or from whom they are received being the corresponding Crs. The Cr. side of the same account contains all the same bills, the parties to whom they are paid being the corresponding Drs.

(2) The Bills payable. The Cr.* side of this account contains all the bills issued by the firm, the parties drawing them being the corresponding Drs.; the Dr. side contains the same bills, the parties paying them at maturity being the corresponding Crs., generally bankers.

But if bills payable or receivable are paid without the intervention of a banker, who is the corresponding Cr.? For example, if I take a bill for any given sum, and at maturity present it to the acceptor, and receive payment in cash, where is the proper credit entry? Not to the acceptor, as he has been credited once when he gave the bill; but the bills receivable account must be credited, because when the acceptor was credited, the bills receivable account was debited, and by the new entry to the Cr. side, the account is balanced.

Similarly, if we pay any acceptance in cash, we must not credit our own personal account, which has already been credited when we gave the bill; but we must credit the bills payable account, which was debited with it when we gave the bill.

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* I here begin with Cr., whereas under "Bills receivable" I began with Dr. A bill receivable account begins with Dr., a bill payable with Cr.

BILLS RECEIVABLE.

Dr. Cr

No.	Received	On whose acct. Cr.	F. L.	Drawn by	Upon	Payable at	Time	Due	Amount	When	and how disposed Dr.	F. L.	
	1861												
1	Jan. 8	Smith & Co.	2	Self	Smith & Co.	L. & W. Bank	2 mo., Jan. 4	Mar. 7	72 8 6	Mar. 7	Cash Book	15	72 8 6
2	„ 18	Atkins & Co.	6	„	Atkins & Co.	Do.	1 mo., Jan. 11	Feb. 14	70 0 0	Feb 14.	C. B.	15	70 0 0
									142 8 6				142 8 6

BILLS PAYABLE.

Dr. Cr.

When paid	By whom paid Cr.	F. L.		No.	Accepted	On whose acct. Dr.	F. L.	Payable to	Time	Where payable	Due	Amount
					1861							
Mar. 18	Cash Book	15	45 10 0	1	Jan. 16	J. Martin	8	Drawers	2 mo. Jan. 16	S. P. & S.	Mar. 19	45 10 0
Feb. 26	Do.	15	200 0 0	2	Jan. 24	Wm. Hinds	5	Do.	1 mo. Jan. 24	S. P. & Co.	Feb. 27	200 0 0

TRIAL BALANCE.

31 Mar. 1861.

			F		F	£	s.	d.
				Joint Stock Bank		165		
				Smith & Co.				
				Martin				
				Wilson				
81	5			Hinds				
				Atkins & Co.				
				Topham		234		
100	18			Phillips & Co.				
				Cotton				
				Port Wine		16	5	
				Tea		8	15	
				Linen		1	13	6
245	16	6		Cash				
				Bills Receivable				
				„ Payable				
60	19			Profit and Loss				
				Trade Expenses		63	5	
488	18	6				488	18	6

We have explained the meaning of a Trial Balance. We have yet to consider a General Balance. This is a complete statement of the position of a firm or company; and when presented in the form of a document is termed a Balance Account. An abstract of this balance account is called a "Balance Sheet." This consists of two parts:

(1) A Statement of Assets and Liabilities, showing whether the party is solvent, or insolvent.

(2) A Statement of Profit and Loss, explaining how the above result of solvency or insolvency has been brought about.

The following are specimens of Balance Sheets, taken from the "Money Market and City Intelligence" of "The Times."

The former is from the accounts of a bankrupt estate, and consists only of a statement of Assets and Liabilities; the second is that of a joint-stock bank, and comprises the statement of the Profit and Loss, as well as the Assets and Liabilities.

In the bank account, the pounds sterling only have been given, for the sake of brevity.

Specimen of a Balance Sheet.

Dr.

				£	s.	d.
To Creditors on open account				1632	9	6
By Bills payable	58271	11	3			
Less, goods consigned	102	11	10			
				58168	19	5
By Liabilities on bills receivable	9959	11	6			
Expected to be duly met	7977	2	11			
	1982	8	7			
Cash in hand at Joint Stock Bank	1835	4	10			
				147	3	9
				59948	12	8

Cr.

				£	s.	d.
By Cash at bankers held as security against bills receivable discounted				[659	3	9]*
Cash in travellers' hands				142	0	11
By debtors, good				3096	17	8
,, ,, doubtful 843 11 1 estimated at				153	0	0
,, ,, bad 567 5 7						
By Bills receivable, in hand,	2667	2	11			
estimated to produce				1857	16	10
By Property				2348	18	2
By Stock				25569	15	11
				33168	9	6
Less, Creditors under £10	55	11	6			
Rent, paid in full	123	18	4			
				179	9	10
				32988	19	8
Deficiency				26959	13	0
				59948	12	8

thus shewing a probable dividend of 10*s.* or 11*s.* in the pound.

The following Balance Sheet shews (1) what money has been received as paid-up capital, and the reserve fund from previous profits not divided, but retained to meet contingencies, and the amount received from depositors, or customers; (2) what the bank has to shew for all the above money, i.e. either money invested, or spent upon preliminary expenses, or securities for money advanced.

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* Not available as an asset.

Dr.		£			Cr. £
To Capital paid up		300,000	By investments in Government securities, &c.		282,374
Reserve fund	20,000		Freehold premises	40,000	
Half-year's interest at 4 per cent.	400		in occupation of the bank	35,000	
		20,400			75,000
Amount due by bank on current accounts		1,599,373	By bills discounted, loans, &c.		1,321,845
Amount carried to credit of Profit and Loss account,	43,934		By cash in hand, and at call		264,564
Less amount paid to customers for interest on balances	13,794		By preliminary expenses		4,116
		30,140*	By lease and buildings at Charing Cross		2,014
		1,949,913			1,949,913

Profit and Loss account.

Dr.	£		Cr. £
Half-year's current expenses, bad debts, &c.	10,904	By balance of Profit and Loss, brought forward from last half-year	1590
Rebate of interest on bills discounted, not yet due, carried to profit and loss new account	3,815	Ditto for current half-year	28550
Amount written off, preliminary expenses	700		
To dividend at 5 per cent. per annum upon £300,000	7500		
Amount carried to credit of Reserve fund	6600		
Balance carried to Profit and Loss new account	621		
	30,140		30,140

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* This is the most important item, and is in reality the balance between the two sides of the capital account and indicates the gross profit. Had the bank lost more than it had made during the half-year, there would have been an item on the *other* side of the capital account, indicating a loss.

In the above, the Cr. side shows all that there is to dispose of from the gross profit indicated in the capital account, and the balance undisposed of in a previous half-year. The Dr. side explains how it is proposed to dispose of it; after all expenses are paid, and a proper addition is made to the reserved fund there remains £8121; and out of this a dividend of £7500 is paid to the shareholders, being at the rate of 5 per cent. per annum; and the remaining balance is carried forward to next half-year's Profit and Loss account.

UNITS OF VALUE, MEASURE, &c.

In dealing so frequently as we do with the various weights, measures, and moneys in use, we must necessarily have observed that there are certain fixed *standards* of each. An investigation into the circumstances which determined the choice of these standards is not uninteresting.

The particular coins, weights, or measures, which have been chosen as standards, whereby we measure all other quantities, are termed *units*. Thus, of the pound, shilling, or penny, any one may be considered as a unit, though, if they preserve their mutual relation unchanged, the determination of one unit will at once give us the knowledge of all the others.

Similarly, the determination of the length of the yard will give the value of all other units of length, as miles, feet, &c. if their relation to the yard be fixed. And so also the determination of the pound avoirdupois will be found to fix the value of every other standard of weight.

Now it will be found in (p. 245), under the head of Decimal Coinage, that in anything like a perfect system, as the French metrical system, there is no unit chosen but what is derived from some natural standard; and in that system we shall see that all the units of weight, value, &c. are derived from one simple unit of length, which again is derived from a measurement accurately taken upon the earth's surface.

But in England, few things can be imagined less influenced by uniformity of principles than the determination of our several units of value, weight, capacity, &c., and their multiples and submultiples.

Thus, the relations of farthings, pence, shillings, and pounds, which involve the multiplication by 4, 12, 20, proceed upon no scientific principle; and the same may be said of nearly all the weights and measures, especially in Dry Measure, and Avoirdupois Weight.

The penny is the real unit among our coins, all others being multiples of it, except the farthing, which is a fraction of it. The recent bronze coinage has completely overthrown the relation which has hitherto been supposed to subsist between the intrinsic value of the metal and its current value; and may probably lead to very important results, in the further substitution of a currency which has rather the nature of a token than an embodiment of value. Copper Coinage is a legal tender only up to 12*d*. Silver Coinage only up to 40*s*.

The standard silver coin is $\frac{37}{40}$ pure. From 1 lb. Troy of this metal are coined 66*s*., so that the mint price of standard silver is 5*s*. 6*d*. per oz., and that of pure silver $=\frac{40}{37}$ of 5*s*. 6*d*. $=$ 5*s*. $11\frac{13}{37}d$.

The standard gold coin is $\frac{11}{12}$ pure. From 1 lb. Troy of this metal are coined $46\frac{29}{40}$ sov. $=$ £46 14*s*. 6*d*.; so that the mint price of standard gold is £3 17*s*. $10\frac{1}{2}d$.; and that of pure gold is $\frac{12}{11}$ of 77*s*. $10\frac{1}{2}d$. $=$ £4 4*s*. $11\frac{5}{11}d$.

A *Carat* is $\frac{1}{24}$ part of 1 lb. of gold: hence the fineness of gold is often expressed by saying that it is so many *carats fine*, meaning so many parts out of 24; thus our standard gold is said to be 22 carats fine, and jewellers' gold is 18 carats fine.

The following coins are worth notice, since they occur in ancient documents:

Groat $=$ 4*d*. Tester $=$ 6*d*. Noble $=$ 6*s*. 8*d*. Angel $=$ 10*s*.
Merk, or Mark $=$ 13*s*. 4*d*. Jacobus $=$ 25*s*.

The adoption of the unit of length is indeed traceable to scientific principles. Thus it is obtained from the length of the seconds' pendulum, oscillating in vacuo at the latitude of London. This length is determined as accurately as possible by experiment, and measured—and when this is divided into 39·13921 equal parts, 36 of them form our yard. And all other measures of length are either multiples of it, as the fathom, pole, furlong, mile, league; or sub-multiples, as the foot, inch, barleycorn. This last is obsolete in practice; and an inch is either divided decimally into tenths, hundredths, &c., or into 12 equal parts called *lines*.

It is worth enquiring whence come the two different measures of a degree, viz. 60 geographical miles, or $69\frac{1}{4}$ British miles. A geographical mile is $\frac{1}{60}$th part of a degree of longitude at the equator, but a British mile $=1760$ yds.; and on comparing these two miles we find that 60 of the former are equal to about $69\frac{1}{2}$, or, in round numbers, 70 of the latter. The advantage of having a geographical mile as a unit, is that it is a convenient length to take as a knot or nautical mile, which knot bears an ascertainable ratio to the length of a degree of longitude, though that ratio varies in different latitudes.

Nautical miles, or knots, can be readily converted into British miles, or the converse; for since 60 knots=70 B. miles; $\therefore$ 1 knot=$\frac{7}{6}$ mile=$1\frac{1}{6}$ miles; and 1 mile=$\frac{6}{7}$ knot=$(1-\frac{1}{7})$ knots; so that to convert knots into miles we add $\frac{1}{6}$th of the number; and to convert miles into knots we take off $\frac{1}{7}$th of the number.

For estimating long periods the year is taken as the unit of time; and for shorter periods, the day. The length of the day may be taken from two or three sources, which give results slightly different. These are as follows:

(1) The interval between any star's leaving the meridian of a place and returning to it. This interval is found never to vary, and is termed a *sidereal day*. It would not, however, be convenient as the ordinary measure of time; for its commencement, depending upon the presence of some particular star upon the meridian, would not be marked by a sufficiently striking phenomenon, and in the course of a year would happen when the Sun was at various distances above and below the horizon.

(2) The time of the Sun's leaving the meridian of any place and returning to it; i.e. the interval between two successive noons; and this interval is termed a *solar day*. But as the Sun's apparent motion in the heavens is not uniform, this solar day varies, and is therefore not a good standard.

(3) The third and most suitable standard is the interval which would be occupied by the Sun, if the equalities of its motion were corrected, that is, if it travelled at the same uniform rate as it now does on an average. This interval is termed the *mean solar day*. A clock which shows a change of 24 hours in such a day is said to keep true, or mean solar time; but the time indicated by it will almost always be different from that shown by a sundial, on account of the Sun's irregular motion; and hence we observe in the almanacs that the clock is sometimes said to be before, and sometimes after the Sun. In fact the clock and the sundial agree only four times in a year.

When therefore we say that the year consists of so many days, we speak of mean solar days. And having thus settled the length of a day, we divide it into 24 equal parts, and call each an hour; divide the hour into 60 equal parts, and call each a minute; divide a minute into 60 equal parts, and call each a second. We then take a pendulum of such a length that it will oscillate in a vacuum, in the latitude of London, 60 × 60 × 24 times, or 86400 times, in one day: and we thus determine the length of the seconds' pendulum.

The length of the year itself is an important unit of time. This is found by observing the interval between the Sun's leaving an equinox or solstice, and returning to the same equinox or solstice. This interval is termed the *Tropical Year*, and is found most accurately by comparing observations made at distant periods. It is found to be 365d. 5h. 48m. 47·58091 secs. mean solar time. As it would be very inconvenient for civil purposes that a year should consist of a broken number of days, (for we should in that case sometimes have a part of a working day in one year, and the remainder in another) it is agreed to take 365¼ days as a year; and to reckon 365 days as a year for 3 consecutive years, and then with the 4 quarters that have been accumulating in 4 years, make an extra day; so that every 4th year consists of 366 days. This day is called an *intercalary* day, and is added to the shortest month, viz. February, causing what is commonly termed Bissextile* or Leap Year.

By observing, however, the difference between the true value of the tropical year, 365·24222, and the assumed value, 365·25, we find there is an excess of 11 min. per annum, which amounts to about three days in 400 yrs. Hence, if we omit three leap years in 400 yrs., we shall very nearly rectify the error. Now every year, divisible by four, is ordinarily a leap year; hence every year that closes a century, since it is divisible by 4, should be a leap year; thus, 1600, 1700, 1800, 1900, &c. should all be leap years. But as we must omit three leap years in 400 yrs., it is agreed that three out of the above four years which terminate successive centuries shall not be leap years. To determine which are the three, we divide by 4 the above numbers without the final ciphers; and we find that 16 can be divided by 4 without remainder, but 17, 18, 19 cannot; hence 1600 is made a leap year, and 1700, 1800, 1900 are not.

The above error of 11 minutes per year went on accumulating from the institution of the Julian calendar till 1582 when the alteration was introduced by Pope Gregory XIII, from whom it was called the Gregorian calendar. Now, the vernal equinox, which in 325, at the Council of Nice, fell on the 21st of March, in 1582 happened on the 11th of March; therefore, Pope Gregory caused 10 days to be omitted, making the 15th of October immediately succeed the 4th; so that the next year the vernal equinox again fell on the 21st of March. This correction was not introduced into England till 1751, when it had become necessary to omit 11 days in the current year.

The time before the alteration is termed *Old Style*, and after it *New Style*. And as in Old Style the year began at the 25th of March, any event which occurred between January 1st and March 25th, would be referable to both years, and was thus noted, Jan. 17th, $16\frac{41}{42}$, or 1641-2. But with the change of style the year was made to commence at the 1st of January.

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* From *bis*, twice, and *sextilis*, sixth; because in this year the *sixth* day before the calends of March was repeated twice.

Russia still adheres to the Old Style, and the error now amounts to about 12 days.

The standard of weight is a lump of platinum, marked P.S. 1 lb. 1844, and called the parliamentary standard pound, and preserved in the Exchequer. This is the pound Avoirdupois, and $\frac{1}{7000}$th of this is termed a *grain*. 5760 of these grains make what is termed a pound Troy. Formerly, there was a lump of bronze, preserved as a standard of the pound Troy, and of course $\frac{1}{5760}$th of it was a grain. But this standard was destroyed in the burning of the Houses of Parliament in 1835. From the above avoirdupois lb. all the other multiples and submultiples are derived. Apothecaries retail by the troy pound and troy grain, but they subdivide the pound in a different way.

The standard of fluid measure is a gallon; and the true definition of it is that it contains 10 lbs. avoirdupois of distilled water weighed in air at a temperature of 62^0 F. when the barometer is at 30 inches.

The weight of a cubic inch of water, expressed in grains, can be best obtained from the following data:

3·937079 Eng. inches = 1 decimètre,
1 cubic decimètre = 1·000012 kilogrammes,
1 kilogramme = 15432·349 grains.

Hence, the weight of a cubic inch of water is found to be 252·5977 grains.

$$\text{Also, since 1 grain} = \frac{1}{252\cdot5977}\text{ cub. in. of water,}$$

$$\text{and 1 gallon, or 10 lbs.} = 10 \times 7000 \text{ grains,}$$

$$1 \text{ gallon} = \frac{70000}{252\cdot5977}\text{ cub. in.} = 277\cdot72 \text{ cub. in.}$$

DECIMAL COINAGE.

THE question of Decimal Coinage, which seemed to interest the public mind some time since, appears now to have lost much of that interest; but it is not lost sight of by any who desire to see the facilities for education extended, or commercial intercourse simplified.

From what has been said above, (p. 240), about our present coinage, it is manifest that there has been no attempt made to proceed upon any system in the construction of our coinage, weights, and measures; but that the choice of every unit has been the result either of accident or arbitrary selection. And though there is primâ facie evidence of the superiority of a decimal over any other system of coinage, seeing that it substitutes simple arithmetic for compound; yet it is not till we come to the consideration of the weights and measures, that we see, to the full extent, how cumbrous our system is, if it deserves the name of system at all. But when we observe the almost infinite variety of weights and measures in use in various parts of the country, especially in the article of grain, so that a merchant who wished to buy or sell by the bushel would have to take a different standard of weight for a bushel in almost every considerable corn market in England, one would think that our present position would hardly find a defender.

The only system of decimal coinage which has been brought extensively and authoritatively under the eye of the British public, is termed the pound and mil system, which was recommended before a Committee of the House of Commons, in 1853, by many able witnesses, among whom were Professors Airy and De Morgan.

But the system which has by far the strongest recommendation for simplicity and completeness is the metrical system of France, which, as we shall presently see, embraces coinage, and weights and measures, linear, superficial, solid and liquid.

The advantages of *any* decimal system of money, weights and measures, in simplifying the operations of Arithmetic, are as follows.

I. There would be no such thing as Compound Arithmetic.

II. Reduction would no longer be treated as a separate and very troublesome process, because all reduction to a lower name would be performed by merely moving the point to the right, and all reduction to a higher by moving the point to the left.

III. There would not be the same necessity for the acquirement of dexterity in the use of Vulgar Fractions, since they would for almost all business purposes be superseded by Decimals*.

IV. The rule called Practice would be entirely abolished; since Practice is only another method of working examples in Compound Arithmetic; and all questions hitherto wrought by it would be worked by Simple Multiplication.

V. Proportion and its numerous applications would of course partake of the great simplification arising from the rejection of all the present processes of Reduction and Compound Rules. The calculations of Interest, Stocks and Exchange, would offer the most noticeable instances of abbreviation.

Probably, under a decimal system, the fluctuations in the price of Stock would be reckoned by *tenths*, instead of as now by *eighths;* so also the brokerage would be £$\frac{1}{10}$, and not £$\frac{1}{8}$, per cent. And these two changes would very materially increase the simplification.

So that probably the whole subject of Arithmetic would be taught in half the time that it is now; and much greater accuracy would be generally obtained.

The pound and mil system of coinage is briefly as follows.

It adopts the present pound, as the unit, and its tenths, hundredths, and thousandths.

Thus, one *tenth* of £1 is the present *florin,*
one *hundredth* of £1 would be a new coin, called a *cent,*
and one *thousandth* of £1 would be a new coin, called a *mil;*
so that 10 mils = 1 cent, 10 cents = 1 florin, 10 florins = 1 pound.

Other coins would be used, but their value on paper must all be expressed in the above four coins, called *money of account.*

Also, in writing a sum of money consisting of pounds, and sums below £1, we should place a dot or point (·) after the pounds, and put the florins in the first place after the dot, the cents in the second, and the mils in the third.

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* All the theory of decimals will of course remain unaltered; and it will still be advisable that pupils should learn the whole system of decimals, including the converting of Vulgar Fractions into Decimals and the converse; Circulating Decimals, &c. But in consequence of the general tendency to supersede Vulgar Fractions by decimals, these parts of the subject may be viewed not so much as a *practical* acquisition, but rather as an instrument of *educational* discipline, an important function of Arithmetic very much overlooked by many, both teachers and learners.

Thus, £23, 4 florins, 5 cents, 7 mils, would be written £23·457; and if each of these quantities be written separately, in four sums under one another, they would stand thus:

£
23.
·4
·05
·007

where in the third line a cipher is placed, to keep the 5 cents in the second column after the point, *i. e.* in the column of cents; and in the fourth line, two ciphers are placed, so as to keep the 7 mils in the column of mils. But in 23·457, the 4 keeps the 5 in its proper column, and the 45 keeps the 7; and therefore no ciphers are required.

But a sum of 10 mils, or more, being equal to a cent, will appear in the second place, and 100 mils or 10 cents, being really a florin, will fill the first place.

Thus, 17 mils = 1 c. 7 m., or in terms of £1 = £·017
75 cents = 7 fl. 5 c. ,, ,, = £·75
And, £·457 may be read 4 fl. 5 c. 7 m.
or 4 fl. 57 mils.
or 457 mils.

This system would have the following coins; those below the sixpence, or quarter florin, are new.

					£	£
Sovereign	=	1000 mils, or written in pounds			1·000 or	1.
Half Sovereign	=	500	...	...	·500 ,,	·5
Crown	=	250	...	...	·250 ,,	·25
Florin	=	100	...	...	·100 ,,	·1
Half Florin or Shilling	=	50	...	...	·050 ,,	·05
Quarter Flo. or Sixpence	=	25	...	...	·025	
Two Cent Piece	=	20	...	...	·020 ,,	·02
Cent	=	10	...	...	·010 ,,	·01
Five Mil Piece	=	5	...	...	·005	
Two Mil	=	2	...	...	·002	
One Mil	=	1	...	...	·001	

In discussing the question of Decimal Coinage, our attention cannot but be forcibly directed to a due consideration of the Metrical System of France, which, on account of its simplicity, completeness and beautiful arrangement, stands preeminent among all other systems.

It is called the *Metrical* System, because from the single unit, the Mètre, are derived the weight and diameter of its coins, and all its weights, and its measures, whether linear, superficial, or cubic. Of course it is plain that in a system so simple and complete as this, all arithmetical operations are performed by means of the four Elementary Rules; and there is no such thing in all French computations as Compound Arithmetic.

The Mètre was obtained from the subdivision of a meridian line drawn upon the earth's surface. By the actual measurement of an arc (from Dunkirk to Barcelona) a computation was made of the length of a quadrant, or one-fourth part of a complete meridian. A ten-millionth part of this quantity was taken as the unit of linear measurement, and is equal to about 39·371 English inches.

All *multiples* of this, proceeding tenfold *higher*, are named from the mètre and *Greek* prefixes, as deca, hecto, kilo, representing 10, 100, 1000 times the mètre. And all *submultiples* of the mètre, proceeding tenfold lower, are named from the mètre and *Latin* prefixes, as deci, centi, milli, representing $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ of the mètre. Hence we have the following table of the values of the French mètre, and its multiples and submultiples, expressed in terms of an English inch.

Myriametre	or	10000	metres	=	393710	English inches.
Kilometre	,,	1000	,,	=	39371·	,,
Hectometre	,,	100	,,	=	3937·1	,,
Decametre	,,	10	,,	=	393·71	,,
METRE	,,	1	,,	=	39·371	,,
Decimetre	,,	$\frac{1}{10}$	,,	=	3·9371	,,
Centimetre	,,	$\frac{1}{100}$	,,	=	·39371	,,
Millimetre	,,	$\frac{1}{1000}$	,,	=	·039371	,,

Here it can hardly be needful to remark, that every one of these denominations can be converted into the next lower, by simply moving the point to the *right, i. e.* by *multiplying* by 10 for each step in the reduction; and to the next higher, by moving the point to the *left, i. e.* by *dividing* by 10 for each step.

It will next be most convenient to discuss the weights. The unit is a Gramme, and as in the mètre, so the multiples of the Gramme proceeding tenfold higher each step are called decagramme, hectogramme, kilogramme; and the submultiples, tenfold lower, are called decigramme, centigramme, milligramme.

The Gramme is the weight of the volume of distilled water, at the temperature of its greatest density, contained in a cube whose edge is a centimetre; and therefore, of course

the weight of 10 such cubes forms a decagramme,
" 100 " " hectogramme,
" 1000 " " kilogramme.

And in like manner

the weight of $\frac{1}{10}$ of such a cube forms a decigramme.
" $\frac{1}{100}$ " " centigramme.
" $\frac{1}{1000}$ " " milligramme.

A gramme is equivalent to 15·432 grs. or nearly 16 grs. Troy,
∴ a kilogramme " 2·68 lbs. or about 2⅔ lbs. Troy.

Again, the volume of a kilogramme of water is taken as the unit of capacity, and is called a Litre: and its multiples, proceeding tenfold higher, are, upon the same principle as before, termed a decalitre, hectolitre, kilolitre; and the submultiples, proceeding tenfold lower, are called a decilitre, centilitre, millilitre.

A litre is equivalent to 1·760773 pts. or about 1¾ pints,
∴ a kilolitre " 220·09668 gallons, or about 220 galls.

The principal unit of superficial measure is the *are* or square decamètre, *i. e.* a square having 10 mètres or linear units for its side.

$$\text{And since } 1 \text{ sq. mètre} = \left(\frac{39{\cdot}371}{36}\right) \text{sq. yds.} = 1{\cdot}196046 \text{ sq. yds.}$$

$$\text{Hence } 1 \text{ are} = 100 \text{ sq. mètres} = 119{\cdot}6046 \text{ sq. yds.}$$
$$\text{a decare} = 1196{\cdot}046.$$
$$\text{a hectare} = 11960{\cdot}46.$$

This last will be found to be from 2⅓ to 2½ acres English, and is commonly employed in France for measuring large pieces of land.

The coins are also derived from the mètre both as to their weight and measure. There are but two moneys of account, the centime, and the franc = 100 centimes; or, considering the centimes as decimal parts of a franc, the name of but one coin is required. Thus, 25 francs, 37 centimes may be written 25·37 francs.

The centime is a coin the diameter of which is 1 centimetre, and weight is 1 gramme; so that 100 centimes touching one another and arranged in a straight line would extend one metre; and treated as weights, they would be equal to 100 grammes, or one-tenth of a kilogramme. Hence, every centime forms at the same time, a coin, a weight, and a measure.

The various coins that are needed and generally used to avoid carrying a multitude of smaller coins are, in silver, $\frac{1}{5}$ franc = 20 cents, $\frac{1}{4}$ franc = 25 cents, $\frac{1}{2}$ franc = 50 cents, 2 franc and 5 franc pieces: and in gold, 5 franc, 10 franc, 20 franc, and 40 franc pieces. Their weights and diameters are as follows:

SILVER.

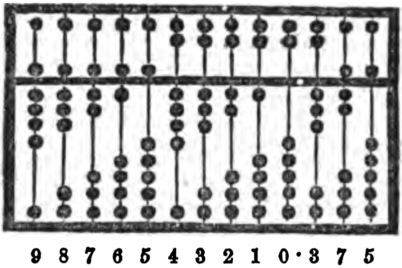
francs.	wt. in grammes.	diam. in millimetres.
$\frac{1}{5}$	1	15
$\frac{1}{4}$	1·25	15
$\frac{1}{2}$	2·5	18
1	5.	23
2	10.	27
5	25.	37

GOLD.

5	1·6129	17
10	3·2258	19
20	6·4516	21
40	12·9032	26
50	16·129	28
100	32·258	35

THE ABACUS.

The *Abacus* or *Swan-pan*, as it is called in China, is an instrument for the quick performance of the operations of the four Simple Rules. As it is applicable only to *Simple* Addition, Subtraction, &c., it cannot be used for commercial purposes, excepting where a decimal coinage is current. The annexed cut represents the instrument. The number of parallel wires is optional: I have inserted ten at the left of the decimal point, and three after it, so that we can count as high as 1000 millions of the unit, and as low as thousandths of the same unit.



Each ball or counter in the upper part of the frame stands for *five;* and each one below for *one*. When any one of the counters is used, it must be brought close to the cross bar which divides the frame; when the balls touch the outer bars, they are out of use for the time. The diagram will shew the number of balls required to represent any one of the digits from 0 to 9; the balls now in use correspond to the numbers below the frame.

In adding together any number of quantities, the operator performs the addition as each line is read over to him[*]; so that when the whole of the rows of figures have been read, the position of the counters after his last operation will give the total of all the sums that were to be added. And since it is said that in China the operator can perform the additions as rapidly as any one will ordinarily read the various amounts, and with perfect accuracy, there must be a considerable saving of time.

---

[*] The pupil should learn to add, beginning from the left, because all sums in money will generally be read over from left to right.

The following principles must be remembered, and the corresponding operations performed with facility.

I. To add *ten* to any row is to carry *one* to the next row.

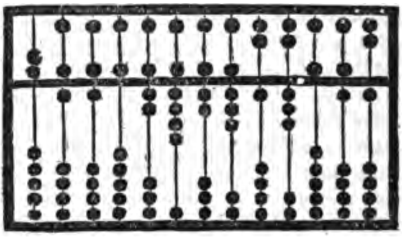
II. If two *fives* are in use on any row, they must be rejected, and *one* carried to the next row.

III. If *five* counters are in use on any bar in the lower frame, it is better to reject them, and substitute a single one in the upper frame.

In performing additions, when there are counters, not in use, sufficient to perform the addition, it is very easy; thus, I can add 2, 3, or 4 to 1, or to 5: or, I can add 6, 7, or 8, to 0 and 1.

But the number of counters will often be too small, especially when we are adding to numbers which already require nearly all the counters to be in use. Thus, I cannot *at one process* add 3 to 8, or to 9, or to 4. In this case, instead of adding 3 directly, I add its equivalent, 5—2, or 10—7, *i. e.* I add 5 and subtract 2, or add 10 and subtract 7.

When the number to which I am adding is *more than five*, I add the larger equivalent, as 10—7, instead of 5; or 10—8, instead of 2: but if it be *less than five*, I add the smaller equivalent, as 5—2, instead of 3; or 5—3, instead of 2.



9876543210·375
786862774·527
2384276·421
1000·000
1·007
6532·485
18·746
10665797813·561

We have seen that the first diagram represents the number 9876543210·375. The second gives the result of the annexed sum in Simple Addition. Each row is added singly, according to the directions given above; and if the work is performed correctly, the operator will find the arrangement of the balls as in the adjoining diagram. I have used numbers involving three places of decimals, in order to show the application of the instrument to a decimal currency. The operation of Subtraction will of course be just opposite to that of Addition: thus, where there are balls enough to enable me to subtract directly, as 2 from 4, or 6, or 8, I do so: otherwise, I must subtract indirectly, as was done in

Addition. To subtract 4, *i. e.* 5—1, from 5, 9, 7, or 8, I subtract 5, and add 1: to subtract 8, *i. e.* 10—2, from any number less than 8, I subtract 10, (that is, 1 from the previous row,) and add 2. And so on for other numbers.

Multiplication by the numbers 2 to 12, or by composite numbers, whose factors are not greater than 12, is easily performed; of course each line, as it is produced mentally, is recorded on the abacus, and takes the place of the previous one. The same may be done with any multiplier, as 257, if the top line or multiplicand be written on a slip of paper, and kept before the eye: for of course, in such an example, if I commence by marking the top line upon the instrument, the arrangement is changed as soon as I commence the multiplication; and for working the second and third line I must have the multiplicand written before me.

Short Division may be very readily performed, but not Long Division; and therefore only the numbers from 2 to 12, and those which are composed of factors not greater than 12, can be used as divisors.

Those who have learnt the Multiplication Table as far as 20 times 20, can of course extend their use of the abacus accordingly.

Obs. In the construction of the abacus, it would be well to have the balls of a different colour after every three rows, reckoning from right to left, in order to assist the eye in catching quickly the place of one thousand, one million, &c.

APPENDIX.

FRACTIONAL QUOTIENT.

IT was mentioned in (22) that the word quotient is not inapplicable, where one number is to be divided by another, and yet the division cannot be performed; thus in the division of 2 by 5, we said that the quotient was $\frac{2}{5}$.

This may be seen more clearly, if we remember that the word *quotient* means *how often* the divisor is contained in the dividend,—and if we observe the following illustration. Suppose that I have a rod five inches long, and with it I am to measure the depth of water in several full vessels, of which the depth is known to be 1, 2, 3, 4,...... inches. If now I ask *how many* times the length of this rod is contained in the depth of the water; the reply is, that in the first, one inch or one-fifth of the rod was immersed, *i. e.* the rod *went into* the depth one-fifth times; and therefore the genuine quotient of 1 when divided by 5 is one-fifth or $\frac{1}{5}$. Similarly, the quotient in the case of the second, third, &c. vessels, would be $\frac{2}{5}$, $\frac{3}{5}$, &c. And when I come to measure the depth of the five-inch vessel, the quotient ought to be $\frac{5}{5}$, or 1, *i. e.* the rod is just once immersed. And when I measure the seven-inch depth, I find the quotient to be $\frac{7}{5}$ or $1\frac{2}{5}$; *i. e.* it takes one immersion of the rod, and $\frac{2}{5}$ of another, to measure the seven inches.

ABSTRACT AND CONCRETE.

An *abstract* number is a number of repetition, or the amount of repetition a certain number goes through; but a *concrete* number is the number which thus goes through the process of repetition.

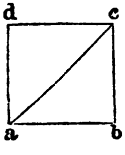
In the process of multiplication, the number that follows the sign is of course by the above definition, abstract, unless we can show that the order of multiplication is immaterial. A concrete number multiplied by a concrete number is inconceivable; when it appears to occur, as in the case 3 ft. × 2 ft. = 6 sq. ft., the expression would be more accurately written 3 sq. ft. × 2 = 6 sq. ft. See *Art.* 178.

Division is the converse of Multiplication, and may be defined as the art of finding how often one quantity is contained in another. It is of two kinds: 1st, where two concrete quantities of the same kind are given, and it is required to find how many times one must be repeated to obtain the other: here, $\frac{\text{concrete dividend}}{\text{concrete divisor}}$ gives as quotient an abstract number, *i.e.* so many times. 2ndly, where an abstract number is given as the divisor, and a concrete number, as before, for the dividend; here the question takes the shape "What number requires to be repeated as many times as the divisor indicates, in order to become the dividend?" And the corresponding definition is, Division is the process of finding what number it is that requires to be multiplied by the divisor to produce the dividend. The resulting number may be the same in both cases, the results differing only in kind: thus, we should have $\frac{35 \text{ lbs.}}{5 \text{ lbs.}} = 7$ times; and $\frac{35 \text{ lbs.}}{5} = 7$ lbs.

RATIO.

It was observed in (174) that quantities which cannot be accurately represented by numbers, are said to be incommensurable. And since the Exs. in Ratio and Proportion, which we have considered, have always involved only commensurable quantities, it might be thought that the application of the principles of Ratio and Proportion was limited to such quantities.

But by observing (66), from whence our definition of Ratio was taken, we learn that a Ratio can exist between any two quantities whose magnitude can be represented, as *A* and *B* are, *i.e.* geometrically: and since it can be shown[*] that magnitudes which cannot be represented accurately in numbers, can yet be correctly represented geometrically; therefore a ratio can exist between two or more quantities, even if one or all of them be incommensurable, though the value of that ratio cannot be accurately represented in numbers. Thus, in Fig. 1, page 172, the ratio of the diagonal of a square inch to its side is represented by the fraction $\frac{a \text{———} c}{a \text{———} b}$; or arithmetically, $\frac{1 \cdot 4142 \ldots\ldots}{1}$ or $1 \cdot 4142 \ldots\ldots$



[*] Some knowledge of Geometry is necessary to enable us to find the length of the line *ac*.

Since *abc* is one angle of a *rectangular* figure, or a *right angle*, we learn from Euclid, I. 47, that the square of which *ac* is the side = the sum of the squares of which *ab* and *bc* are the sides. Now, in Fig. 1, p. 172, we took $ab = 1$, and $ac = 1$; therefore the sum of the squares of *ab* and $ac = 1^2 + 1^2 = 2$; therefore the square of $ac = 2$, or *ac* itself $= \sqrt{2} = 1 \cdot 4142 \ldots$

CIRCULATING DECIMALS.

Sometimes a decimal of very long period may be easily carried out to many places, without performing the division throughout, as in the following Ex.

To reduce $\frac{1}{19}$ to a decimal.

$$\begin{array}{r} 19)\ 1{\cdot}00\ ({\cdot}05263 \\ \underline{95} \\ 50 \\ \underline{38} \\ 120 \\ \underline{114} \\ 60 \\ \underline{57} \\ \underline{\underline{3}} \end{array}$$

By division we have

$$\frac{1}{19} = {\cdot}05263\frac{3}{19} \qquad \text{(H)}$$

therefore, multiplying both sides by 3,

$$\frac{3}{19} = {\cdot}15789\frac{9}{19}$$

and substituting for $\frac{3}{19}$ in (H) we have

$$\frac{1}{19} = {\cdot}0526315789\frac{9}{19};$$

therefore $\frac{9}{19} = {\cdot}4736842101\frac{81}{19} = {\cdot}4736842105\frac{5}{19}$;

and hence $\frac{1}{19} = {\cdot}052631578947368421 05\frac{5}{19}$; and by continuing this process, it is plain that we double at every step the number of figures previously obtained. This decimal, it will be seen, circulates after the eighteenth figure; so that $\frac{1}{19} = {\cdot}\dot{0}5263157894736421\dot{1}$.

PROPORTION.

The pupil who is acquainted with Book VI. of Euclid's Ele-

FIG. 9.

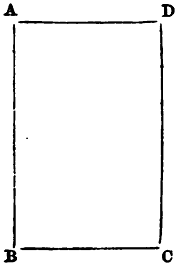
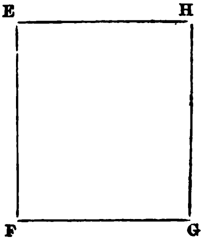


FIG. 10.



ments of Geometry, will know that if two rectangular figures, AC, EG, be equal, the four sides which contain a pair of the equal angles, ABC, EFG, are proportional in the following order,—AB, EF, FG, BC; so that

$$AB : EF :: FG : BC \qquad \text{(J)}.$$

We can show that this proportion is such as would be obtained from the statement of a question in Rule of Three, in which two surfaces AC, EG, are required to be equal; and until we know of this proportion, we have no right to employ the ordinary process of Simple Proportion.

Ex. How wide a piece of cloth, 15 feet long, will cover a floor 13 feet 6 inches long and 10 feet wide?

Let AB be the length = 15 feet; BC the breadth, which is yet to be found; EF = 13 ft. 6 in.; FG = 10 ft.; then, by the usual statement, and writing BC as the fourth term, we have

$$15 \text{ ft.} : 13 \text{ ft. } 6 \text{ in.} :: 10 \text{ ft.} : BC;$$

and this will be found to be the same as (J).

In the language of Geometry, we say that the sides about the equal angles ABC, EFG, are *reciprocally* proportional.

The solution of such a question as the following is worth notice.

The shadow of a steeple is 105 yards long, and that of a stick 4½ feet long is 15 yards: find the height of the steeple.

FIG. 11.

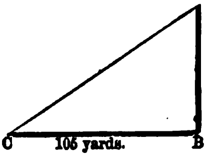
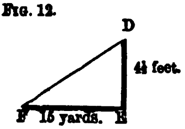


FIG. 12.



Let AB, BC, represent the steeple and its shadow; and let DE, EF represent the stick and its shadow: I have to find the length of AB.

Now, most pupils, seeing this question under the head of Rule of Three, would immediately take it for granted that the three terms given would form a statement; *i. e.* that the ratio between the lengths of the two shadows is equal to that of the steeple and stick by which those shadows are cast. This is quite true; and my object in explaining this sum is merely to show what authority we have for believing that these two ratios are equal. Join AC, DF,—these lines AC, DF, will represent the direction of rays of light from the sun, and being from the same distant body are considered parallel: hence, since AB, DE, are parallel, as are also FE, CB; the triangles, ABC, DEF, are said to be similar: and from a geometrical property of such triangles, we have the following:

$$EF : BC :: DE : AB;$$

or, substituting the value of EF, BC, DE, which are given by the question, we have

$$15 \text{ yards} : 105 \text{ yards} :: 4\tfrac{1}{2} \text{ feet} : AB;$$

and therefore $AB = \dfrac{105 \text{ yards}}{15 \text{ yards}} \times 4\tfrac{1}{2} \text{ ft.} = 31\tfrac{1}{2}$ feet.

The following Ex. will show that it is not always safe to assume that a question which is apparently a Rule of Three Ex. will at once furnish a statement; *i. e.* that two quantities which appear to be connected, as in ordinary Exs., are really proportional to one another, as in (75) and (76).

Ex. If a ball falling from rest drop through a space of $64\frac{2}{5}$ feet in two seconds, through what distance will it have fallen in three seconds?

Now, if it were true that the distance fallen in any time were *directly proportional* to the time, the statement would be

$$2 \text{ seconds} : 3 \text{ seconds} :: 64\tfrac{2}{5} \text{ feet},$$

and the fourth term would be $96\frac{3}{5}$ feet. But a knowledge of the laws of mechanics corrects this supposition, and teaches us that the distance fallen in any time is not directly proportional to the number of seconds, but to the *square* of that number; and that in the above statement I ought to have put 2^2 and 3^2, or 4 and 9, instead of 2 and 3: hence the statement is

$$4 : 9 :: 64\tfrac{2}{5} \text{ feet};$$

and the distance fallen in three seconds $= 144\frac{9}{10}$ feet.

EXCHANGE.

To show that £1 = 25·17 francs, nearly.

£1 of English gold 22 carats fine is $\frac{11}{12}$ pure, or contains $91\frac{2}{3}$ gold in 100 parts. £1 of French gold contains 90 parts in 100. 1 kilogramme is coined into 155 Napoleons, or 3100 francs; therefore 1 kilogramme of English gold is worth 3157·4 francs; or deducting the charge for coining, namely 6·7 francs per kilog., £1 Eng. gold = 3150·7 francs. Also 1 oz. Troy, or 480 grains = 31·1 grammes, at the rate of 15·432 grains to 1 gramme. The Mint price of standard gold in England = 77*s.* $10\frac{1}{2}$*d.* per oz., because 1 lb. Troy is equivalent to $44\frac{1}{2}$ guineas, or $934\frac{1}{2}$ sh.; therefore 1 oz. is worth $934\frac{1}{2}$*s.* ÷ 12, or 77·875*s.*

Collecting these facts, we have

$$\text{£}1 = 20s.$$

$$77{\cdot}875s. = 1 \text{ oz. Troy}$$

$$1 \text{ oz. Troy} = 31{\cdot}1 \text{ grammes},$$

$$1000 \text{ grammes of Eng. gold} = 3150{\cdot}7 \text{ francs};$$

$$\therefore \text{£}1 = \frac{20 \times 31{\cdot}1 \times 3150{\cdot}7}{77{\cdot}875 \times 1000} \text{ francs} = 25{\cdot}17 \text{ francs, nearly.}$$

MISCELLANEOUS QUESTIONS.

ON FRACTIONS AND THE PRINCIPLES OF PROPORTION.

1. **Explain** the terms "Common Measure," and "Common Multiple."

2. Show how to find the Least Common Multiple of any set of whole numbers. Take any numbers as an Example.

3. When are numbers said to be prime to one another? Give an example of three numbers which are prime to one another, but none of them primes.

4. What do you mean by prime *factors*? Resolve 1512 into its prime factors.

5. Show what is the G. C. M. of 27, 144, 96, by breaking them into their prime factors.

6. Explain the terms *proper* and *improper* fractions. Give an Ex. of each.

7. State the two modes of multiplying fractions by whole numbers. Take an Ex., and show that the modes are true.

8. State also the two modes of *dividing* fractions by whole numbers, and prove their correctness in any example.

9. What do you mean by reducing a fraction to lowest terms? Show the correctness of the process employed.

10. Taking any Ex. of the above reduction, draw a figure which shall enable us to see by inspection that the fraction when so reduced remains unaltered *in value*.

11. For what purpose do we require the L. C. M. of any numbers?

12. Show how to reduce improper fractions to mixed numbers, and the contrary; giving an example of each kind of fractions.

13. What must be done with fractions before they can be added or subtracted? Take as Exs. $\frac{3}{4}+\frac{5}{6}$; $\frac{7}{8}-\frac{3}{5}$.

14. How do you compare two proper fractions, at sight, so as to ascertain which is the larger? Ex. Compare $\frac{7}{11}$ and $\frac{9}{13}$.

15. Prove that $\frac{5}{7}\times\frac{2}{3}=\frac{10}{21}$; and that $\frac{3}{5}$ of $\frac{2}{7}=\frac{6}{35}$.

16. Hence show that *of* and the sign (×) placed between two fractions have the same meaning.

17. Show that the common rule for division of fractions, viz. "In-

vert the divisor, and proceed as in multiplication," is correct. Ex. Prove that $\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \times \frac{7}{2}$.

18. What do you understand by the brackets in the following Ex. $(\frac{3}{5} + \frac{2}{7}) \times (\frac{2}{3} + \frac{3}{4})$?

19. Explain the process of exhibiting any fractional quantity in positive terms. Ex. Express $\frac{7}{8}$ of 27*s.* in shillings, pence, and fractional parts of a penny.

20. What is the meaning of the term RATIO? How do you express the ratio of 4 to 5, and of 5 to 4?

21. Give distinctive names to the ratios 2 : 5 ; 3 : 3 ; 5 : 2.

22. Exhibit the ratio of 3*s.* to 7½*d.* as an abstract number.

23. What is Proportion? The test of proportionality among 4 quantities is whether or no the product of the extremes equals the product of the means. Prove this.

24. Explain the reason of the following operations in Rule of Three; 1st, The reduction of the 1st and 2nd terms to the same name: 2nd, The multiplying of the 2nd and 3rd terms together and dividing by the 1st.

25. Explain how the term *proportional* is used to embrace two kinds of proportion. Give an Ex. of each.

26. Give a description of the mode of working the following questions:

(1) Reduce 3*s.* 6*d.* to the fraction of £1.

(2) What fraction of a half guinea is equivalent to a moidore?

27. Work the following Ex., and be particular in forming a correct fractional quotient in the pence. Ex. (£8 17*s.* 9¼*d.*) ÷ $6\frac{7}{8}$.

28. Show how to compare fractional quantities. Ex. Compare $\frac{3}{5}$, $1\frac{2}{7}$, $\frac{5}{9}$.

29. Compare also $\frac{5}{7}$£, $\frac{7}{8}$ of a guinea, and $\frac{5}{8}$ of 16*s.* 8*d.*

30. If $\frac{7}{15}$ths of a piece of work were done in 1 hour, how soon would all the work be done? What relation is there between the work done in 1 hour, and the time of doing all the work?

ON DECIMALS.

1. Explain the term "Decimal Fractions;" and show the principle upon which we write tenths, hundredths, thousandths, &c. decimally.

2. Show how to convert a *vulgar* fraction into a *decimal* fraction, written as a vulgar fraction.

3. Explain what is meant by a power of any number; and write 16, divided by the 4th power of 10, (1) as a vulgar frac^{n}; (2) as a decimal.

4. State the method of converting into a decimal, a fraction which contains a power of 10 as a denominator.

5. How do you multiply a decimal by a power of 10? Ex. $3{\cdot}75 \times 10^4$.

6. How do you divide a decimal by a power of 10? Ex. $18{\cdot}76 \div 10^5$.

7. State what fractions produce *terminating* decimals, and what produce *non-terminating*. Explain the reason.

8. If the den^r of a fr^n, in its lowest terms, which produces a circulating decimal, be known, what may be known concerning the length of the period of the decimal? Illustrate by the Ex. $\frac{3}{7}$.

9. Show how to convert terminating decimals into vulgar fractions. Ex. $\cdot 0605$.

10. Explain fully the mode of converting non-terminating decimals into vulgar fractions. Take as Exs. $\cdot\dot{3}4\dot{3}$, $\cdot 36\dot{7}2\dot{5}$.

11. State the mode of adding together decimals—(1) terminating, (2) non-terminating.

12. State the mode for subtraction.

13. Show the truth of the common rule for the multiplication of decimals. Ex. Find the value of $1{\cdot}75 \times \cdot 037$.

14. Explain the mode of performing Long Division in Decimals; and show what varieties may occur in fixing the decimal point in the quotient. Give Exs. in illustration.

15. Show how to perform the operations of multiplication and division of circulating decimals.

16. Find the value of $3{\cdot}756 \times 21{\cdot}9875$, correct to 2 places of decimals: and explain the principle and mode of working by the contracted form of multiplication in decimals.

17. Find the product of $13{\cdot}0586$ by $12{\cdot}758$, without either assuming the rule for multiplication of decimals, or converting them to vulgar fractions.

18. Give Exs. of reduction of decimals,
   (1) From a decimal of £1 into positive terms, as shillings, pence, &c.
   (2) From a quantity involving pounds, shillings, and pence, to the decimal of a moidore. Work both your Exs.

ON PRACTICE.

1. What do you mean by the term "aliquot parts"? Give Exs.

2. What is the highest aliquot part of £1? What of 1s.?

te out tables of aliquot parts, (1) of 1s. (2) of £1.

4. Find by Practice the cost of 7108 lbs. at $9\frac{3}{4}$*d*. Explain all the work.

5. Write down one of every variety of Examples in Practice.

6. Explain the mode of finding the cost of $5605\frac{6}{7}$ articles at any price.

7. Show also how to find the value of 1105 yds. at 7*s*. $6\frac{9}{10}$*d*.

8. Give an Ex. in Practice in which the usual mode of taking aliquot parts of the highest denomination in money cannot be employed. Show how the Ex. is to be worked.

ON PROPORTION.

1. How many terms are generally found in proportion?

2. What is the general object of a question in Rule of Three?

3. In stating a Rule of Three sum, which term do you write down first, and how do you select it?

4. When the statement is completed, describe the remaining steps of the work.

5. Write down the fractional form of the fourth term, in terms of the other three; and hence derive the rules for completing a sum after it is stated.

6. From the fractional form mentioned in the last question, shew that the fourth term will be of the same nature as the third term.

7. Give an example in Rule of Three involving four terms, of which one will not appear in the statement.

8. Show how three terms are to be obtained from the following Ex. What is the length of a floor which is 16 feet broad, and equal in area to a floor 24 feet square?

9. What is the difference between Simple and Compound Proportion?

10. Prove that one statement may be made to produce the same result as two or more separate statements by Single Rule of Three. Construct an Example in illustration.

ON THE APPLICATIONS OF PROPORTION.

1. Explain the terms *Principal, Rate, Interest, Amount.*

2. Distinguish between Simple and Compound Interest.

3. What other questions come properly under the head of Simple Interest?

4. Prove that the following Rule for finding Simple Interest is true: "Multiply the sum given by the rate per cent., and divide by 100."

5. Show how to find the amount of any sum of money when out at interest for any number of years and days.

6. Write in a plain Rule of Three form the questions involved in the following Examples:

(1) At what rate per cent. will £175 4*s.* 2*d.* amount to £196 4*s.* 8*d.* in 3 years?

(2) In what time will £175 amount to £204 15*s.*, at $4\frac{1}{4}$ p. c. per annum?

(3) What sum will amount to £155 5*s.* in 3 years at 5 p. c. Simple Interest?

7. What do you mean by so many years' purchase being given for a piece of property? Construct and work an Ex. involving the above phrase.

8. Show the difference between Interest and Discount.

9. Explain whether the debtor or creditor is benefited by counting discount as interest.

10. What is meant by discounting a bill?

11. State and explain the technical terms connected with such bills.

12. If a tradesman wishes to throw off any required discount off the invoice price of any article, as 20, 30, &c. per cent., show what proportion of the net value must be added, so as to give a proper gross price.

Ex. I. Find the invoice price of an article worth 21*s.*, so as to allow of a discount of 15 per cent.

Ex. II. If $\frac{2}{7}$ of a net price be put on, to make the gross price, what discount is it intended to throw off?

13. Explain the objects proposed in questions under the head of "Profit and Loss."

14. How do you in all examples introduce the condition of losing 10 per cent., or of gaining 10 per cent.?

15. Write down a rule for working examples in FELLOWSHIP or Partnership,

(1) When the time is the same.

(2) When the time is different.

16. Show how to divide any sum of money into parts proportional to any given numbers. Ex. Divide £325 into shares proportional to the numbers 2, 1, $\frac{1}{2}$.

17. Construct an Ex. concerning the mixing together of articles of the same species, but of different values; and find the value of any quantity of the mixture.

18. State and explain the terms used in the buying and selling of STOCKS.

19. Place in a Rule of Three form the following questions:

(1) What must be given for £2520 stock at $87\frac{7}{8}$, and 2*s*. 6*d*. per cent. commission?

(2) What per centage will be obtained by investing in the 3 per cents at $96\frac{1}{8}$, allowing $\frac{1}{8}$ per cent. for commission?

(3) How much stock can be bought for £1450, when the price is $104\frac{1}{8}$, and commission 2*s*. 6*d*. per cent.?

20. What are the principal questions that arise under the head of EXCHANGE?

21. Explain the terms *Par of Exchange, Course of Exchange*, and *Simple* and *Compound Arbitration*.

22. Explain the mode of payment for imports and exports between America and England, by means of Bills of Exchange.

23. What is meant by the Chain Rule? Construct an Ex. which will exemplify its use, and work it both with and without that rule.

24. What is the object of the Rule termed Equation of Payments? Explain the opposite views adopted in solving questions under this head.

25. Work the following question without using any statement:

If 12 yards of cloth, 3 qrs. wide, cost £19, what will be the cost of 8 yds. 5 qrs. wide?

AREA AND VOLUME.

1. Describe the several meanings of the term inch, foot, or yard, when used in 1, 2, and 3 dimensions.

2. Define the term *rectangular*.

3. Give the shapes and names of the areas formed by the following products:

(1) 1 foot × 1 foot.
(2) 1 inch × 1 foot.
(3) 1 inch × 1 inch.
(4) 1 inch × $\frac{1}{12}$ inch.
(5) $\frac{1}{12}$ inch × $\frac{1}{12}$ inch.

4. State the mode of finding the number of square feet, &c. in a rectangular area.

5. Show how the Tables called "Square Measure," and "Solid Measure," have been formed.

6. Show by actual computation of the area of a slab, 3 ft. 7 in. by 2 ft. 9 in., that the process given by Cross Multiplication is correct.

7. Give the shapes and names of the solids formed by the following products:

1 sq. foot × 1 linear inch.
1 sup. pr. × 1 linear inch.
1 sq. inch × 1 linear inch.

8. Show by actual computation of the volume of a block, 4 ft. 6 in. long, 2 ft. 4 in. wide, and 1 ft. 9 in. thick, that the process given by Cross Multiplication is correct.

9. Find the value of the above volume by multiplying together the above dimensions, as fractional parts of a foot.

10. Explain the Gunter's Chain; and show how to convert an area expressed in links into acres, &c.

11. What is meant by an incommensurable quantity? Give an example of such a quantity.

EXTRACTION OF ROOTS.

1. Explain the terms INVOLUTION and EVOLUTION.

2. What is meant by the following expression $\sqrt[5]{18}$?

3. Explain the terms perfect square, cube, &c.

4. What will be the form of a square root of a number which is not a perfect square?

5. Give a name to such quantities as $\sqrt{2}$, $\sqrt[3]{5}$.

6. Explain the mode of *pointing*, previous to the extraction of the square root, both of whole numbers and decimals.

7. Show how to find a multiplier which shall make any proposed number a perfect square, or a perfect cube.

Ex. Find the multiplier which shall make 45, (1) a perfect square, (2) a perfect cube.

8. Write down the squares of all the digits.

9. Give a rule for the extraction of the square root.

10. Show why the incomplete divisor does not always give a correct figure in the root.

11. How do you find any power of a fraction? How do you extract the root of a fractional quantity?

12. Give two methods of extracting the square root of a fractional quantity, of which the denominator is not a perfect square.

13. Give a rule for the extraction of the Cube Root.

14. Give a rule for the extraction of any root whatever.

15. Extract the square root of 576, and the cube root of 1728, in an algebraical form.

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